



# **The Objective Indefiniteness Interpretation of Quantum Mechanics:**


**Partition Logic,  
Logical Information Theory,  
and Quantum Mechanics**

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# Outline of the argument: I

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- Quantum mechanics incompatible with usual Boolean logic of subsets where elements definitely have a property or not.
  - But there is a recently-developed dual logic, the logic of partitions about indefinite properties that can be made more definite.
  - Obvious thing to check is if QM is compatible with that other logic, dual to ordinary subset logic.
  - *If* QM was *also* incompatible with the other logic, then one is authorized to push the panic button and:
    - abandon interpretation with Copenhagen instrumentalist approach,
    - cling to Boolean logic by postulating a hidden reality of definite properties with the Bohmians, or
    - soar off in the void with exotica like the "many worlds" interpretation with the Everettians.

# Outline of the argument: II

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- But QM fits perfectly with partition logic.
  - Key concept is old idea of objective indefiniteness. Partition logic (including logical information theory) and lifting program provide the back story so that old idea then gives the *objective indefiniteness interpretation of QM*.
  - This interpretation provides the back story to the *standard view* of a quantum state:  
superposition = *complete* description.
  - I.e., Indefiniteness of a superposition is objective.



# The Logic of Partitions: the dual to the Boolean logic of subsets.

Why was partition logic only recently  
developed?

# "Propositional" logic $\Rightarrow$ Subset logic

- *"The algebra of logic has its beginning in 1847, in the publications of Boole and De Morgan. This concerned itself at first with an algebra or calculus of classes, ... a true propositional calculus perhaps first appeared...in 1877." [Alonzo Church 1956]*
- Variables refer to *subsets* of some universe  $U$  (not propositions) and operations are *subset* operations.
- Valid formula ("tautology") = result of substituting any subsets for variables is the universe set  $U$  for any  $U$ .
- Boole himself noted that to determine valid formulas, it suffices to only take subsets  $\emptyset = 0$  and  $U = 1$ .

# Duality of Subsets and Quotient Sets



- Tragedy of 'propositional' logic = propositions don't dualize so the concept of a dual logic was missing.
- Subsets do have a dual, namely, partitions.
- Category Theory gives subset-partition duality: "*The dual notion (obtained by reversing the arrows) of 'part' [subobject] is the notion of partition.*" (Lawvere)
- A *set partition* of a set  $U$  is a collection of subsets  $\pi = \{B, B', \dots\}$  that are mutually disjoint and the union is  $U$ .
- A *distinction* or *dit* of a partition  $\pi$  is an ordered pair  $(u, u')$  of elements in different blocks  $B \neq B' \in \pi$ .

# Table of Dual Logics

	Subset Logic	Partition Logic
'Elements'	Elements $u$ of a subset $S$	Distinctions $(u,u')$ of a partition $\pi$
All 'elements'	Universe set $U$	Discrete partition <b>1</b> (all dits)
No 'elements'	Empty set $\emptyset$	Indiscrete partition <b>0</b> (no dits)
Duality	Subsets are images $f()$ of injections $f:S \rightarrow U$	Partitions are inverse-images $f^{-1}()$ of surjections $f:U \rightarrow T$
Formula variables	Subsets of $U$	Partitions on $U$
Logical operations	$\cup, \cap, \Rightarrow, \dots$	Partition ops. = Interior of subset ops. applied to dit sets
Formula $\Phi(\pi, \sigma, \dots)$ holds at 'element'	Element $u$ is in subset $\Phi(\pi, \sigma, \dots)$	Pair $(u, u')$ is a distinction of partition $\Phi(\pi, \sigma, \dots)$
Valid formula $\Phi(\pi, \sigma, \dots)$	$\Phi(\pi, \sigma, \dots) = U$ for any subsets $\pi, \sigma, \dots$ of any $U$ ( $ U  \geq 1$ )	$\Phi(\pi, \sigma, \dots) = \mathbf{1}$ for any partitions $\pi, \sigma, \dots$ on any $U$ ( $ U  \geq 2$ )

# *Review of Symbolic Logic* (June 2010)

## Logic of Partitions paper @ [www.ellerman.org](http://www.ellerman.org)

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### THE LOGIC OF PARTITIONS: INTRODUCTION TO THE DUAL OF THE LOGIC OF SUBSETS

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**Abstract.** Modern categorical logic as well as the Kripke and topological models of intuitionistic logic suggest that the interpretation of ordinary “propositional” logic should in general be the logic of subsets of a given universe set. Partitions on a set are dual to subsets of a set in the sense of the category-theoretic duality of epimorphisms and monomorphisms—which is reflected in the duality between quotient objects and subobjects throughout algebra. If “propositional” logic is thus seen as the logic of subsets of a universe set, then the question naturally arises of a dual logic of partitions on a universe set. This paper is an introduction to that logic of partitions dual to classical subset logic. The paper goes from basic concepts up through the correctness and completeness theorems for a tableau system of partition logic.



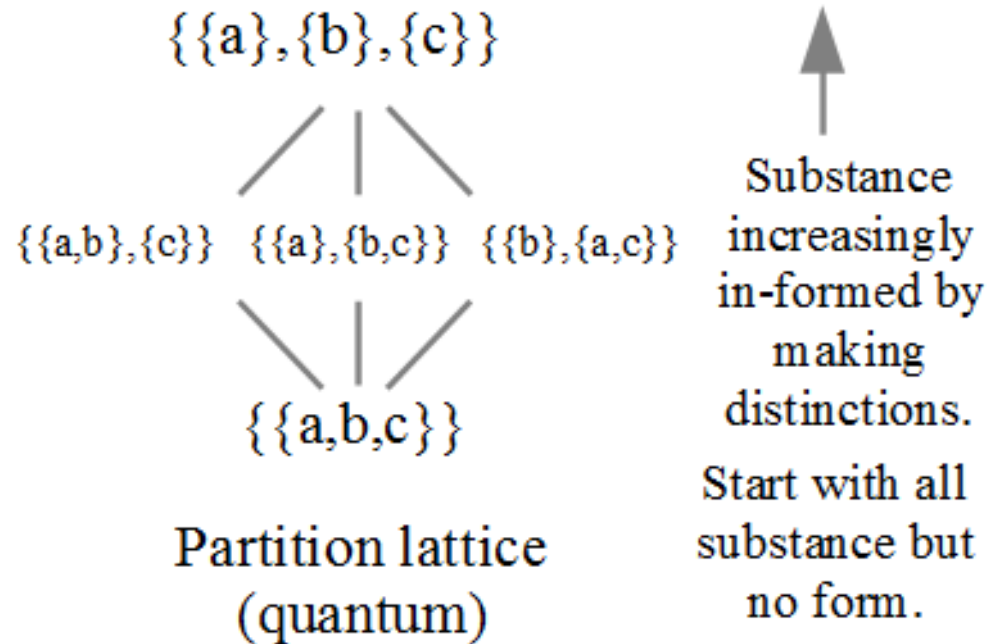
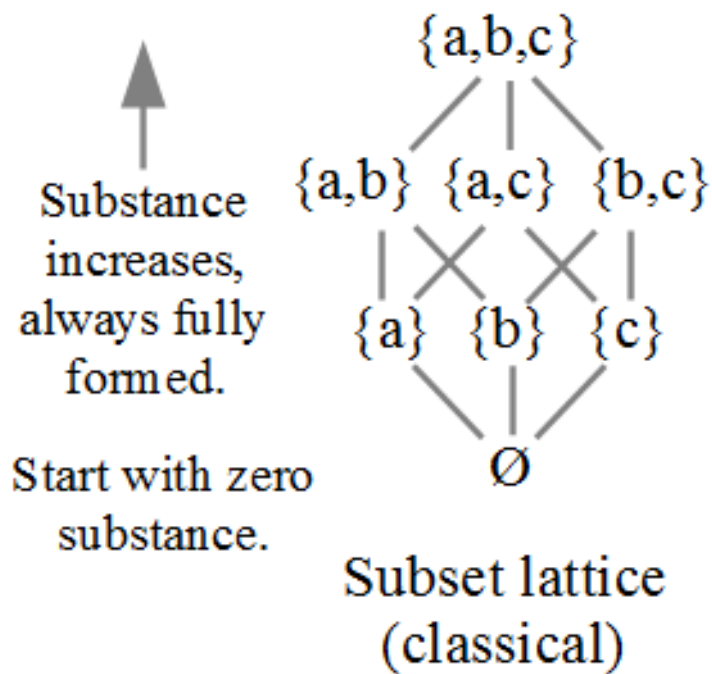


# The Idea of Objective Indefiniteness

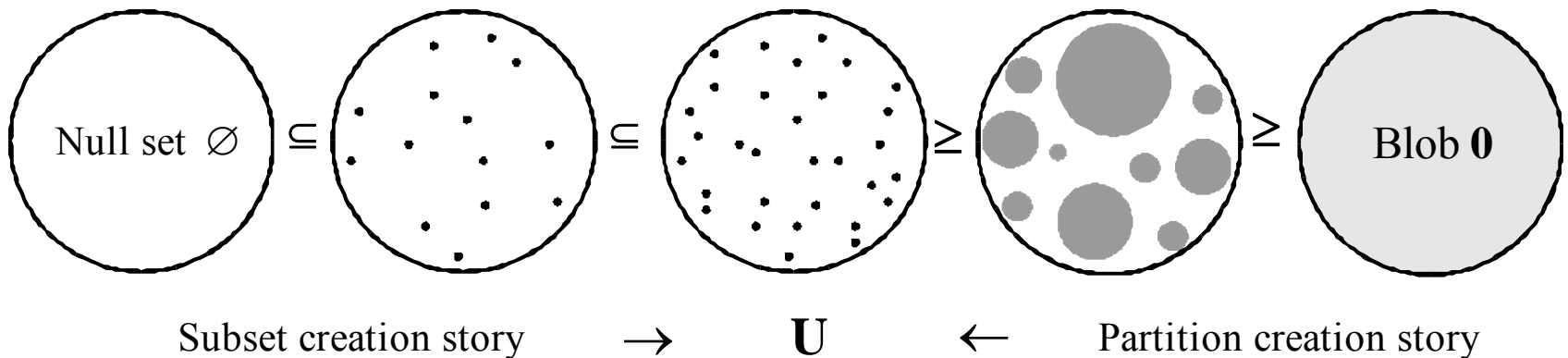
# Partition logic = logic of indefiniteness

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- The new form of logic, partition logic, dual to ordinary Boolean subset logic, gives a new *vision* of micro-reality based on partitions and objectively indefinite entities, and that vision provides the *objective indefiniteness* interpretation of QM.
  - **Basic idea:** interpret block of partition, say  $\{a,b,c\}$ , not as subset of three distinct elements; but as one *indistinct* element that, with distinctions, could be projected to  $\{a\}$ ,  $\{b\}$ , or  $\{c\}$ .
  - Overview of the argument here:
    - the mathematics of partitions using sets can be “lifted” to vector spaces where superposition = vector sum.
    - the result is essentially the math of QM, and hence
    - the micro-reality described in QM *fits* this interpretation.

# Tale of two lattices




# Dual creation stories: 2 ways to create a Universe $U$









- Subset creation story: “*In the Beginning was the Void*”, and then elements are created, fully propertied and distinguished from one another, until finally reaching all the elements of the universe set  $U$ .
- Partition creation myth: “*In the Beginning was the Blob*”, which is an undifferentiated “substance” (with perfect symmetry) and then there is a “Big Bang” where elements (“its”) are structured by being objectively informed (objective “dits”) by the making of distinctions (e.g., breaking symmetries) until the result is finally the singletons which designate the elements of the universe  $U$ .
- In sum, to reach  $U$  from the beginning:
  - increase the size of subsets, or
  - increase the refinement of quotient sets.

# Old idea of objective indefiniteness

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- Basic idea of *objective indefiniteness* is not new:
    - "objectively indefiniteness" (Shimony)
    - "incompletely determined state" (Mittelstaedt)
    - "inherent indefiniteness" (Feyerabend).
    - "unsharp quantum reality" (Busch & Jaeger).
  - Indeed, *standard view* = superposition description is complete, then indefiniteness is objective.
  - As distinctions are made (e.g., measurements), *objectively indefinite states* are made more distinct.
  - Fully distinct states = *eigen* states.

# Superposition: + =

## How to interpret it?

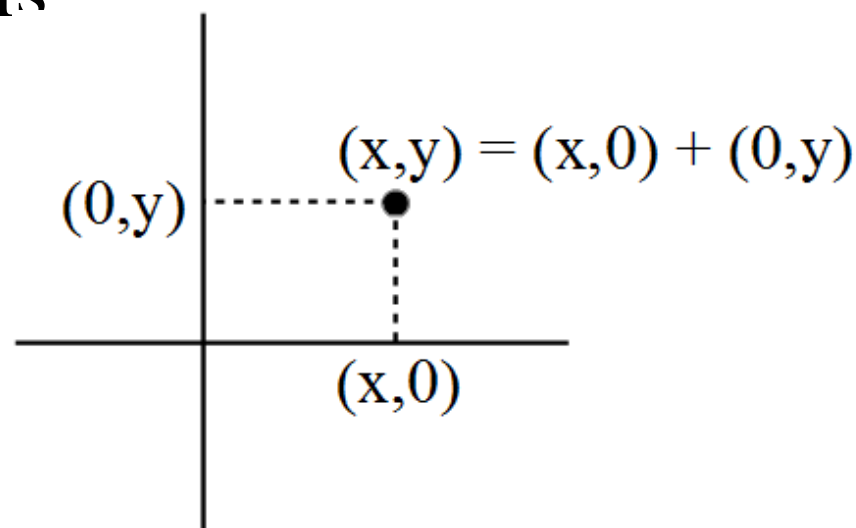
Eigenstate 1: Guy Fawkes with goatee	
Eigenstate 2: Guy Fawkes with mustache	
Objectively indef. superposition: pre-distinction state before facial hair.	
Bohmian superposition: subjectively indef. but objectively one or other is definite.	 or 
But not simultaneously definite like a double exposed photograph.	

# Taking vector spaces seriously

■ Point  $(x,y) = (x,0) + (0,y)$  is **not** simultaneously on x-axis and y-axis.

■ Point  $(x,y)$  is **not** either  $(x,0)$  or  $(0,y)$  but we don't know which.

■ But on projection,  $(x,y)$  becomes definitely  $(x,0)$  or definitely  $(0,y)$ .



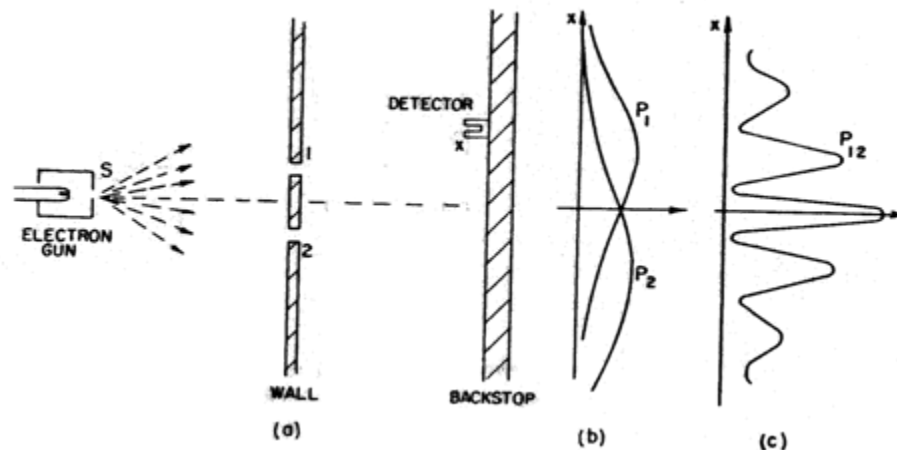
# Wave-particle "duality" = Indefinite-definite particle "duality"

## ■ Example: double-slit experiment.

- No distinguishing between slits → “wave-like aspects” appear (i.e., interference) but
- Distinguish between slits with a measurement (e.g., close a slit or insert detector in a slit) → “wave-like aspects disappear.

## ■ Translation in objective indistinctness interpretation:

- No distinctions → “indistinctness aspects” appear;
- Make distinctions → “collapse of indistinctness.”



- "If you could, *in principle*, distinguish the alternative *final* states (even though you do not bother to do so), the total, final probability is obtained by calculating the *probability* for each state (not the amplitude) and then adding them together. If you *cannot* distinguish the final states *even in principle*, then the probability amplitudes must be summed before taking the absolute square to find the actual probability." [Feynman et al. Lectures Vol. III, p. 3-9]

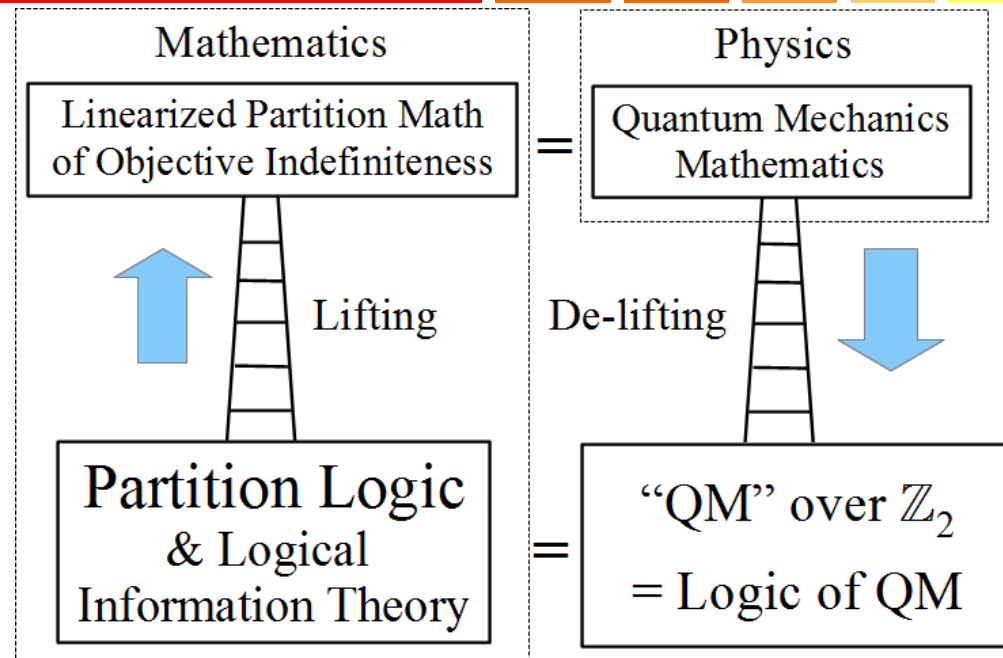




# The Lifting Program:

From Partition Math to  
Vector Space Math


# Linearized Partition Math = QM Math




## Objective Indefiniteness Interpretation of Quantum Mechanics

- **Basis Principle as a conceptual "algorithm"**: Apply a set concept to a basis set and see what concept it generates in the vector space.
- Intuitions can be initially guided using the linearization map  $U \mapsto \mathbb{C}^U$ .
- For instance, apply set concept of cardinality to basis set and get vector space concept of dimension. Cardinality lifts to dimension.

# What is the lift of a *set partition*?

- 
- Concept of basis set is also the vehicle to lift concept of “set partition” to corresponding concept for vector spaces.
  - Take a set partition  $\pi = \{B, B', \dots\}$  of a *basis set* of  $V$ ; the blocks  $B$  generate subspaces  $W_B \subseteq V$  which form a direct sum decomposition of  $V$ :  $V = \sum_{B \in \pi} \oplus W_B$ .
  - Hence a *vector space partition* is defined to be a direct sum decomposition of the space  $V$ .
  - Hence lifting takes direct sum decompositions from sets to vector spaces, not quotient sets to quotient spaces.
  - An earlier proposition-oriented attempt to relate partitions to QM math emphasized *set-partitions* defined by subspaces, i.e., for  $W \subseteq V$ ,  $v \sim v'$  if  $v - v' \in W$ . That wrong-partition approach is called "Quantum logic."

# What is the lift of a *join* of set-partitions?

- 
- Set Definition: Two set partitions  $\pi = \{B, B', \dots\}$  and  $\sigma = \{C, C', \dots\}$  are ***compatible*** if defined on a common universe  $U$ .
  - Lifted Definition: Two vector space partitions  $\omega = \{W_\lambda\}$  and  $\xi = \{X_\mu\}$  are said to be ***compatible*** if they have a common basis set, i.e., if there is a basis set so they are generated by two set partitions on that same basis set.
  - Set Definition: If two set partitions  $\pi = \{B, B', \dots\}$  and  $\sigma = \{C, C', \dots\}$  are compatible, their ***join***  $\pi \vee \sigma$  is defined and is the set partition whose blocks are the non-empty intersections  $B \cap C$ .
  - Lifted Definition: If two vector space partitions  $\omega = \{W_\lambda\}$  and  $\xi = \{X_\mu\}$  are compatible, their ***join***  $\omega \vee \xi$  is defined and is the vector space partition whose subspace-blocks are the non-zero intersections  $W_\lambda \cap X_\mu$  (which is generated by the join of the two set partitions on any common basis set).

# What is the lift of a *set-attribute* $f: U \rightarrow \mathbb{R}$ ?

- If  $f$  is constant on subset  $S \subseteq U$  with value  $r$ , then formally write  $f \upharpoonright S = rS$ , and call  $S$  an “eigenvector” of  $f$  and  $r$  an “eigenvalue.”
- As subsets get smaller, all functions are eventually constant, so for universe  $U$ ,  $\exists$  partition  $S_1, \dots, S_n, \dots$  of  $U$  such that formally:

$$f = r_1 S_1 + \dots + r_n S_n + \dots$$

- For any “eigenvalue”  $r$ , define  $f^{-1}(r)$  = “eigenspace of  $r$ ” as union of “eigenvectors” for that “eigenvalue.”
- Since “eigenspaces” span  $U$ , function  $f: U \rightarrow \mathbb{R}$  is represented by:

$$f = \sum_r r \chi_{f^{-1}(r)}.$$

"Spectral decomposition" of set attribute  $f: U \rightarrow \mathbb{R}$ .

- Therefore an attribute, which is constant on blocks  $\{f^{-1}(r)\}$  of a set partition, lifts to something constant on the blocks (subspaces) of a vector space partition. Spectral decomp.

$$f = \sum_r r \chi_{f^{-1}(r)} \text{ lifts to } L = \sum_\lambda \lambda P_\lambda.$$

# Attributes $f:U\rightarrow\mathbb{R}$ lift to linear operators!


Lifting Program	Set concept	Vector space concept
Eigenvalues	$r$ s.t. $f _S = rS$ for some $S$	$\lambda$ s.t. $Lv = \lambda v$ for some $v$
Eigenvectors	$S$ s.t. $f _S = rS$ for some $r$	$v$ s.t. $Lv = \lambda v$ for some $\lambda$
Eigenspaces	$\{S: f _S = rS\} = \wp(f^{-1}(r))$	$\{v: Lv = \lambda v\} = W_\lambda$
Partition	Set partition of “eigenspaces” $f^{-1}(r)$	Vector space partition of eigenspaces $W_\lambda$
Characteristic functions	$\chi_S:U\rightarrow\{0,1\}$ for subsets $S$ like $f^{-1}(r)$	Projection operators for subspaces like $W_\lambda = P_\lambda(V)$
Spectral decomposition	Set attribute $f:U\rightarrow\mathbb{R}$ : $f = \sum_r r\chi_{f^{-1}(r)}$	Hermitian linear operator: $L = \sum_\lambda \lambda P_\lambda$

# *Compatible partitions $f^{-1}$ , $g^{-1}$ lift to eigenspace partitions of commuting operators*



- Set fact: Join of inverse image partitions of two attributes  $f: U \rightarrow \mathbb{R}$  and  $g: U' \rightarrow \mathbb{R}$  defined iff attributes are compatible, i.e.,  $U = U'$ .
- Vector space fact: Eigenspace partitions of two linear ops  $L$  and  $M$  are compatible so join is defined iff the operators commute, i.e.,  $LM = ML$ .
- Eigenspace partitions of commuting operators  $L, M$  generated by basis of simultaneous eigenvectors that diagonalize the operators.

## Complete joins determine eigen-alternatives

- 
- Set case: Given two same-domain set attributes  $f, g: U \rightarrow \mathbb{R}$ , the blocks  $f^{-1}(r) \cap g^{-1}(s)$  in the join are uniquely labeled by ordered pairs  $(r, s)$  of values, e.g., (age, weight) of people in a room. Set of same-set attributes is *complete* if join of their partitions is discrete (i.e., all 1-element subsets).
  - Vector space case: Given two commuting ops., the blocks  $W_\lambda \cap X_\mu$  in the join of eigenspace partitions are uniquely labeled by ordered pairs of eigenvalues  $(\lambda, \mu)$ . Set of commuting ops. is *complete* (CSCO) if join of eigenspace partitions is nondegenerate (i.e., all 1-dim. subspaces). Unique labels are supplied by ordered set of attribute or observable labels, e.g., simultaneous eigenkets  $|\lambda, \mu, \dots\rangle$ .



# Summary: Lifting from sets to vector spaces

Lifting Summary	Set concept	Vector space concept
Partition	Direct sum decomposition $\pi = \{B\}$ of $U$ : $U = \uplus B$	Direct sum decomposition $\{W_i\}$ of $V$ : $V = \sum \oplus W_i$
Real-valued Attribute	Function $f: U \rightarrow \mathbb{R}$	Hermitian operator $L: V \rightarrow V$
Partition of attribute	Inverse-image partition $\{f^{-1}(r)\}$ for $f: U \rightarrow \mathbb{R}$	Eigenspace partition $W_L = \{W_\lambda\}$ for $L: V \rightarrow V$
Compatible partitions	Partitions $\pi, \sigma$ on same set $U$	Vector space partitions $\{W_i\}$ and $\{X_j\}$ with common basis
Compatible attributes	Attributes $f, g: U \rightarrow \mathbb{R}$ defined on same set $U$	Commuting operators $LM = ML$ , i.e., common basis of simultaneous eigenvectors.
Join of compatible attribute partitions	$f^{-1} \vee g^{-1} = \{f^{-1}(r) \cap g^{-1}(s)\}$ for $f, g: U \rightarrow \mathbb{R}$	$W_L \vee W_M = \{W_\lambda \cap W_\mu\}$ for $LM = ML$
CSCO	Singleton blocks of $\vee f_i^{-1}$ for compatible attributes $\{f_i^{-1}\}$	One-dim. blocks of $\vee W_{L_i}$ for commuting operators $\{L_i\}$



# The Delifting Program:

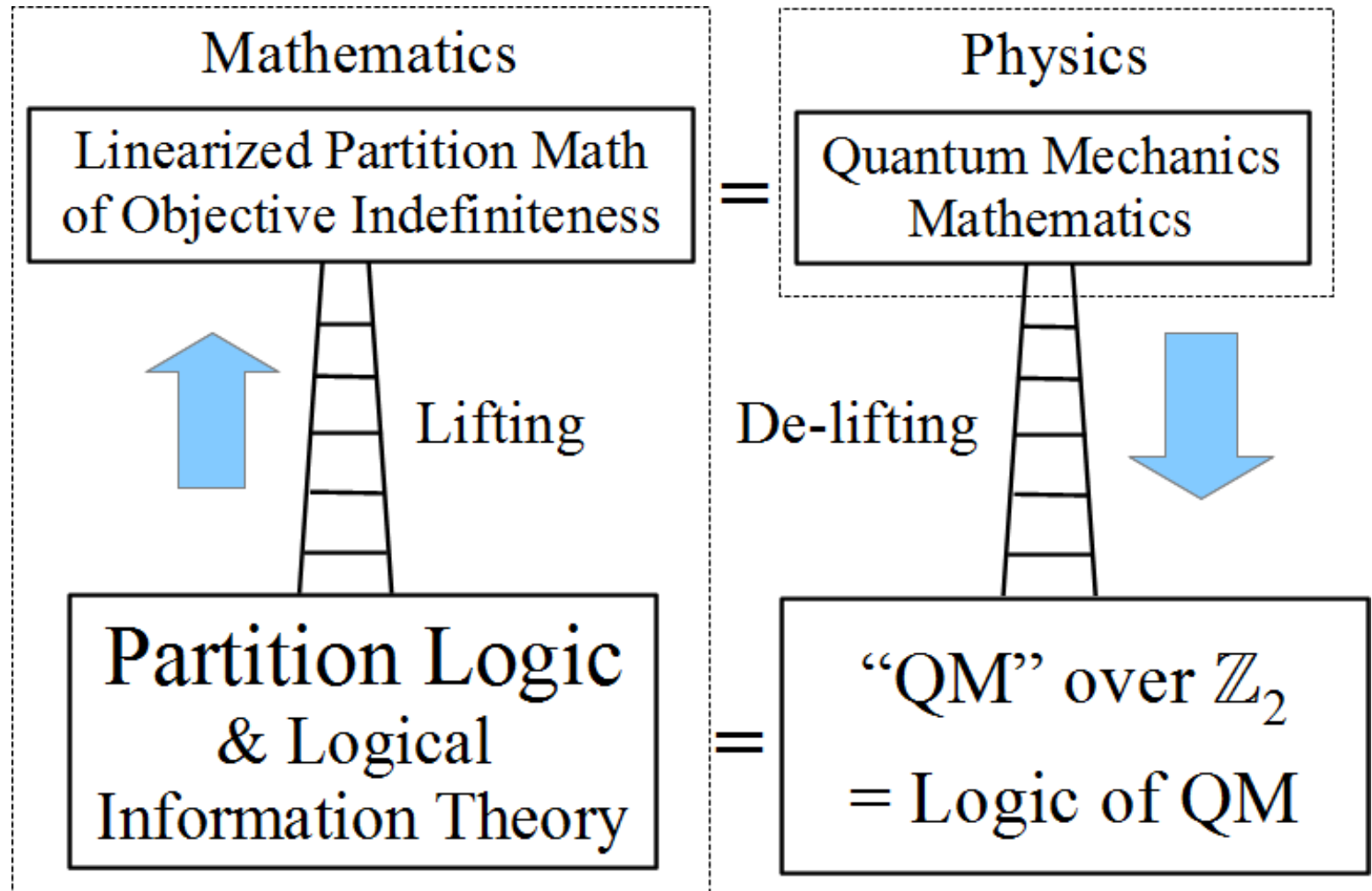
From Vector Spaces back to Sets

"Quantum Mechanics on Sets"

as the (new)

*Logic* of Quantum Mechanics

# Delifting Program: "QM" over $\mathbb{Z}_2$



**Objective Indefiniteness Interpretation  
of Quantum Mechanics**

# Delifting to "Quantum mechanics" over $\mathbb{Z}_2$

- Delifting program: creating set versions of QM concepts to have "quantum mechanics" on sets.
- Key step is conceptualizing  $\wp(U)$  as  $\mathbb{Z}_2^{|U|}$  the  $|U|$ -dimensional vector space over 2, so delift takes base field:  $\mathbb{C} \rightarrow \mathbb{Z}_2$
- Vector addition = symmetric difference of sets:

$$S+T = S \cup T - S \cap T.$$

- Example:  $U = \{a,b,c\}$  so  $U$ -basis is  $\{a\}$ ,  $\{b\}$ , and  $\{c\}$ . Now  $\{a,b\}$ ,  $\{b,c\}$ , and  $\{a,b,c\}$  is also a basis since;  $\{a,b\} + \{a,b,c\} = \{c\}$ ,  $\{b,c\} + \{c\} = \{b\}$ , and  $\{a,b\} + \{b\} = \{a\}$ . Hence take them as singletons of new basis set  $U' = \{a',b',c'\}$  where  $\{a'\} = \{a,b\}$ ,  $\{b'\} = \{b,c\}$ , and  $\{c'\} = \{a,b,c\}$ .

$U = \{a,b,c\}$	$U' = \{a',b',c'\}$
$\{a,b,c\}$	$\{c'\}$
$\{a,b\}$	$\{a'\}$
$\{b,c\}$	$\{b'\}$
$\{a,c\}$	$\{a',b'\}$
$\{a\}$	$\{b',c'\}$
$\{b\}$	$\{a',b',c'\}$
$\{c\}$	$\{a',c'\}$
$\emptyset$	$\emptyset$

- Ket = row table:  $\mathbb{Z}_2^3 \equiv \wp(U) \equiv \wp(U')$

# New Foundations for Quantum Logic = "QM" over $\mathbb{Z}_2$



Set Case: "QM" over $\mathbb{Z}_2$	Hilbert space case: QM over $\mathbb{C}$
Projections $S \cap () : \wp(U) \rightarrow \wp(U)$	$P : V \rightarrow V$
Spectral Decomp. $f \upharpoonright () = \sum_r r (f^{-1}(r) \cap ())$	$L = \sum_\lambda \lambda P_\lambda$
Compl. $\sum_r f^{-1}(r) \cap () = I : \wp(U) \rightarrow \wp(U)$	$\sum_\lambda P_\lambda = I$
Orthog. $r \neq r', [f^{-1}(r) \cap ()] [f^{-1}(r') \cap ()] = \emptyset \cap ()$	$\lambda \neq \lambda', P_\lambda P_{\lambda'} = 0$
Brackets $\langle S _U T \rangle =  S \cap T  = \text{overlap for } S, T \subseteq U$	$\langle \psi   \varphi \rangle = \text{"overlap" of } \psi \text{ and } \varphi$
Ket-bra $\sum_{u \in U}  \{u\}\rangle \langle \{u\} _U = \sum_{u \in U} (\{u\} \cap ()) = I$	$\sum_i  v_i\rangle \langle v_i  = I$
Resolution $\langle S _U T \rangle = \sum_u \langle S _U \{u\} \rangle \langle \{u\} _U T \rangle$	$\langle \psi   \varphi \rangle = \sum_i \langle \psi   v_i \rangle \langle v_i   \varphi \rangle$
Norm $\ S\ _U = \sqrt{\langle S _U S \rangle} = \sqrt{ S }$ where $S \subseteq U$	$ \psi  = \sqrt{\langle \psi   \psi \rangle}$
Pythagoras $\ S\ _U^2 = \sum_{u \in U} \langle \{u\} _U S \rangle^2 =  S $	$ \psi ^2 = \sum_i \langle v_i   \psi \rangle^* \langle v_i   \psi \rangle$
Laplace $S \neq \emptyset, \sum_{u \in U} \frac{\langle \{u\} _U S \rangle^2}{\ S\ _U^2} = \sum_{u \in S} \frac{1}{ S } = 1$	$ \psi\rangle \neq 0, \sum_i \frac{\langle v_i   \psi \rangle^* \langle v_i   \psi \rangle}{ \psi ^2} = 1$
$\ S\ _U^2 = \sum_r \ f^{-1}(r) \cap S\ _U^2 = \sum_r  f^{-1}(r) \cap S  =  S $	$ \psi ^2 = \sum_\lambda  P_\lambda(\psi) ^2$
$S \neq \emptyset, \sum_r \frac{\ f^{-1}(r) \cap S\ _U^2}{\ S\ _U^2} = \sum_r \frac{ f^{-1}(r) \cap S }{ S } = 1$	$ \psi\rangle \neq 0, \sum_\lambda \frac{ P_\lambda(\psi) ^2}{ \psi ^2} = 1$
Born Rule: $\text{Pr}(r S) = \frac{\ f^{-1}(r) \cap S\ _U^2}{\ S\ _U^2} = \frac{ f^{-1}(r) \cap S }{ S }$	$\text{Pr}(\lambda \psi) = \frac{ P_\lambda(\psi) ^2}{ \psi ^2}$
Average of attribute: $\langle f \rangle_S = \frac{\langle S _U f \upharpoonright () _S \rangle}{\langle S _U S \rangle}$	$\langle L \rangle_\psi = \frac{\langle \psi   L   \psi \rangle}{\langle \psi   \psi \rangle}$

Probability math for "QM" over  $\mathbb{Z}_2$  and for QM over  $\mathbb{C}$

# Born Rule = Laplace Rule over $\mathbb{C}$



"QM" over $\mathbb{Z}_2$	QM over $\mathbb{C}$
State = subset $S \subseteq U$ = vector in $\mathbb{Z}_2^{ U }$ represented as: $ S\rangle = \sum_u \langle \{u\}  _U S \rangle   \{u\} \rangle$ for $\langle \{u\}  _U S \rangle \in \mathbb{Z}_2$	State = vector in $\mathbb{C}^n$ represented as: $ \psi\rangle = \sum_i \langle v_i   \psi \rangle   v_i \rangle$ for $\langle v_i   \psi \rangle \in \mathbb{C}$
$\ S\ ^2 = \sum_u \langle \{u\}  _U S \rangle^2 =  S $	$ \psi ^2 = \sum \langle v_i   \psi \rangle \langle v_i   \psi \rangle^*$
Non-degenerate (1-to-1) attribute: $f: U \rightarrow \mathbb{R}$ where $f^1(r) = u_r$ .	Non-degenerate Hermitian op. $L$ with eigenvalues $\{\lambda\}$ and eigen-basis $\{v_\lambda\}$ , $Lv_\lambda = \lambda v_\lambda$ .
Laplace Rule: $\Pr(r S) = \langle \{u_r  _U S \rangle^2 / \ S\ ^2$	Born Rule: $\Pr(\lambda \psi) = \langle v_\lambda   \psi \rangle \langle v_\lambda   \psi \rangle^* /  \psi ^2$

In each case, (absolute) square of coefficient (normalized) is probability.



# Logical Information Theory:

New Foundations  
for Information Theory  
(classical and quantum)

# Normalized counting measures in subset logic and partition logic

	Logical Finite Prob. Theory	Logical Information Theory
'Outcomes'	Elements $u \in U$ finite	Distinctions $(u, u') \in U \times U$ finite
'Events'	Subsets $S \subseteq U$	Dit sets $\text{dit}(\pi) \subseteq U \times U$
Normalized counting measure	$\text{Prob}(S) =  S / U  = \text{logical probability of event } S$	$h(\pi) =  \text{dit}(\pi) / U \times U  = \text{logical entropy of partition } \pi$
Equiprobable outcomes	$\text{Prob}(S) = \text{probability randomly drawn element is an outcome in } S$	$h(\pi) = \text{probability randomly drawn pair (w/replacement) is a distinction of } \pi$

- $\text{dit}(\pi) = \text{set of distinctions [pairs } (u, u') \text{ in different blocks] of } \pi$ .
- Progress of definition of logical entropy:
  - Partitions:  $h(\pi) = |\text{dit}(\pi)|/|U \times U| = 1 - \sum_{B \in \pi} [|B|/|U|]^2$ ;
  - Probability distributions:  $h(p) = 1 - \sum p_i^2$ ;
  - Density operators:  $h(\rho) = 1 - \text{tr}(\rho^2)$ .



## Counting distinctions: on the conceptual foundations of Shannon's information theory

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**Abstract** Categorical logic has shown that modern logic is essentially the logic of subsets (or “subobjects”). In “subset logic,” predicates are modeled as subsets of a universe and a predicate applies to an individual if the individual is in the subset. Partitions are dual to subsets so there is a dual logic of partitions where a “distinction” [an ordered pair of distinct elements  $(u, u')$  from the universe  $U$ ] is dual to an “element”. A predicate modeled by a partition  $\pi$  on  $U$  would apply to a distinction if the pair of elements was distinguished by the partition  $\pi$ , i.e., if  $u$  and  $u'$  were in different blocks of  $\pi$ . Subset logic leads to finite probability theory by taking the (Laplacian) probability as the normalized size of each subset-event of a finite universe. The analogous step in the logic of partitions is to assign to a partition the number of distinctions made by a partition normalized by the total number of ordered  $|U|^2$  pairs from the finite universe. That yields a notion of “logical entropy” for partitions and a “logical information theory.” The logical theory directly counts the (normalized) number of distinctions in a partition while Shannon's theory gives the average number of binary partitions needed to make those same distinctions. Thus the logical theory is seen as providing a conceptual underpinning for Shannon's theory based on the logical notion of “distinctions.”

**Keywords** Information theory · Logic of partitions · Logical entropy · Shannon entropy

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This paper is dedicated to the memory of Gian-Carlo Rota—mathematician, philosopher, mentor, and friend.

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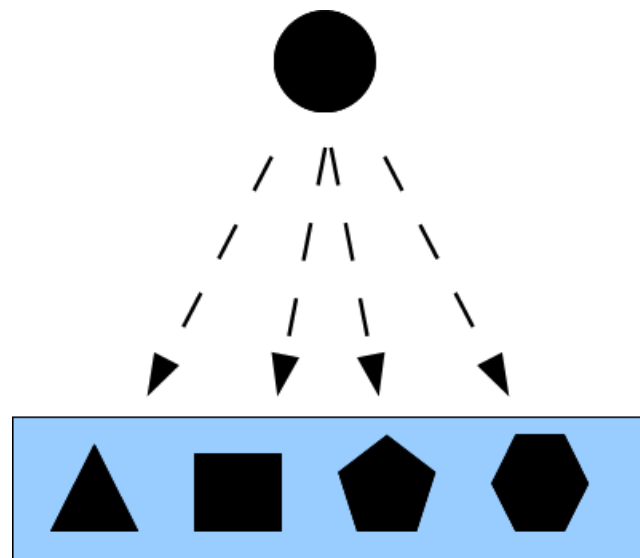
*Synthese* (May 2009)  
Paper on Logical  
Information Theory  
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Philosophically "Deriving" the  
Axioms of Quantum Mechanics  
using the  
Objective Indefiniteness  
Interpretation

# Measurement using Weyl's gratings

- Weyl is one of few quantum physicists to, in effect, use lifting program to go from set partition induced by set attribute to vector space partition or "grating" given by eigenspaces of an operator.
- Measurement applies the grating to the quantum state.
- Indefinite blob of dough falls through one of the polygonal holes with equal probability and then acquires that shape.
- Blob = triangle + square + ... but is not simultaneously all shapes!



# Measurement in "QM over sets"

- $U = \{a,b,c\}$  with real-valued attribute  $f:U \rightarrow \mathbb{R}$  with the "eigenvalues":
  - $f(a) = 1,$
  - $f(b) = 2,$
  - $f(c) = 3.$

- Three "eigenspaces":
  - $f^{-1}(1) = \{a\},$
  - $f^{-1}(2) = \{b\},$
  - $f^{-1}(3) = \{c\}.$

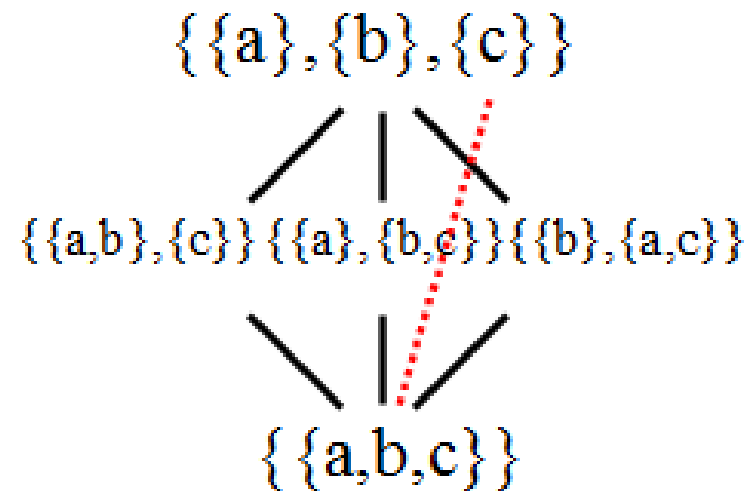
- Take given state  $S = U = \{a,b,c\}.$

- Measurement of observable  $f$  in state  $S$  have probabilities:


$$\Pr(r|S) = |f^{-1}(r) \cap S|/|S| = 1/3 \text{ for } r = 1,2,3.$$

- If result was  $r = 3$ , the state resulting from "projective measurement" is

$$f^{-1}(3) \cap S = \{c\}.$$



# QM over sets: Density matrix = Indit-amplitude matrix

- 
- QM represents state  $S = \{a,b,c\}$  by a density matrix (rows & columns labeled by a,b,c).
  - Each entry like the (a,c) in NE corner is defined:  

$$\rho_{ac} = \sqrt{p_a p_c} \quad \text{if } (a,c) \text{ is an indit, else } \rho_{ac} = 0.$$
  - Thus  $\rho_{ac} = \text{indistinction-amplitude}$  so  $|\rho_{ac}|^2$  is two measurement prob. of getting (a,c) if a,c are indistinct in S.
  - Since all pairs are equiprobable indistinctions for indiscrete partition  $\{U\}$ , density matrix  $\rho$  is all 1/3s.
  - Logical entropy:  $h(\rho) = 1 - \text{tr}[\rho^2] = 0$  since the indiscrete partition is a pure state (no dits).

$$\rho = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

# Non-degenerate measurement



- After non-degen. meas.,  $\hat{\rho} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$  so that the logical entropy is:

$$h(\hat{\rho}) = 1 - \text{tr}[\hat{\rho}^2] = 1 - \left(\frac{1}{9} + \frac{1}{9} + \frac{1}{9}\right) = \frac{2}{3}$$

- Change in logical entropy = sum of squares of indit-amplitudes zeroed in measurement.
- All six off-diagonal terms were zeroed so change in logical entropy is:

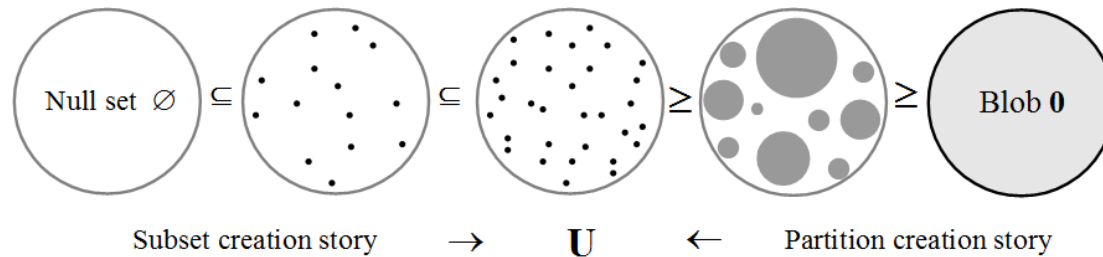
$$6 \times (1/9) = 2/3 = h(\hat{\rho}) - h(\rho)$$

- Dictionary: cohere = indistinct; decohere = distinct; non-degen. meas. = discrete post-measurement partition, i.e.,  $h(\mathbf{1}) = (3 \times 3 - 3) / (3 \times 3) = 6/9 = 2/3$ .

# Logical entropy measures measurement

- In QM, indit-amplitude lifts to coherence-amplitude so for *pure*  $\rho$ ,  $\text{tr}[\rho^2] = \text{sum of two-measurement probs. of getting cohering (i.e., indistinct) eigenstates} = 1$ .
- In QM, a (nondegenerate) measurement turns pure-state density matrix  $\rho$  to the mixed-state diagonal matrix  $\hat{\rho}$  with the same diagonal entries  $p_i$ :
 
$$\rho = \begin{bmatrix} p_1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & p_2 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & p_n \end{bmatrix} \Rightarrow \hat{\rho} = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & p_n \end{bmatrix}$$
- Hence the logical entropy  $h(\rho) = 1 - \text{tr}[\rho^2]$  goes from 0 to  $h(\hat{\rho}) = 1 - \sum_i p_i^2$ .
- For any measurement, the increase in the logical entropy  $h(\hat{\rho}) - h(\rho) = \sum_{i \neq j} |\rho_{ij}|^2 = \text{sum of coherence } (\approx \text{indit}) \text{ terms } |\rho_{ij}|^2 \text{ that are zeroed or decohered by measurement.}$

# Time-independent Schrödinger equation



- Recall from the partition logic creation story, there is some "substance" that is neither created nor destroyed but is structured by distinctions. Hence there will be one observable above all the others, the "amount-of-substance" operator.
- Amount-of-substance attribute lifts to the most fundamental "substance"-operator  $H$  whose eigenstates are stationary:

$$H|E\rangle = E|E\rangle.$$

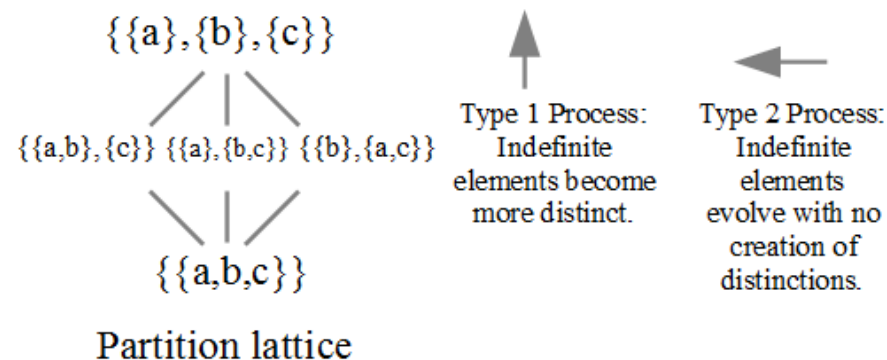
"Substance"-operator eigenvector equation

- Thus obj. indef. interpretation *gives* mathematical form of basic *time-independent Schrödinger equation* that uses the one fundamental operator.
- But the interpretation does not give the form of the operator  $H$  nor the interpretation of "substance"  $E$ . That's physics, not logic.
- Heisenberg: "Energy is in fact the substance from which all elementary particles, all atoms and therefore all things are made..."



# Time-dependent Schrödinger equation

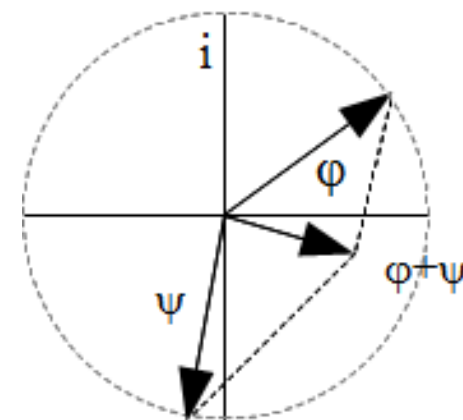
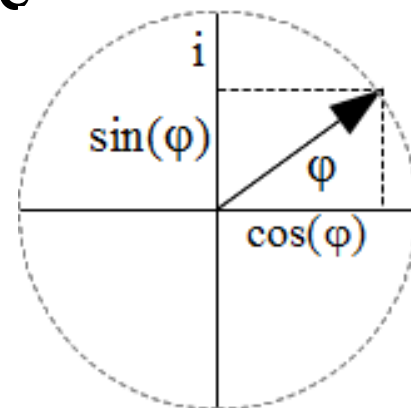
- What about time-dependent Schrödinger equation?
- Measurements make distinctions, so what is the evolution of *closed* quantum system with no interactions that make distinctions?



- What is no-distinctions evolution?
- No-distinctions evolution is evolution with:  
*constant degree of indefiniteness.*
- The degree of indefiniteness or "overlap" between states  $|\varphi\rangle$  and  $|\psi\rangle$  is given by their inner product  $\langle\varphi|\psi\rangle$ .
- Hence the transformations of quantum systems that preserve degree of indefiniteness are the ones that preserve inner products, i.e.,  
*the unitary transformations.*

# Objective Indefiniteness and "Waves"

- Stone's Theorem gives Schrödinger-style "wave" function:  $U(t) = e^{iHt}$ .
- In simplest terms, a unitary transform. describes a rotation in complex space.
- Vector described as function of  $\varphi$  by Euler's formula:  $e^{i\varphi} = \cos(\varphi) + i \sin(\varphi)$ . Complex exponentials & their superpositions are "wave functions" of QM.
- Dynamics = adding rotating vectors.
- Hence obj. indef. interp. explains the "wave math" (e.g., interference & quantized solutions) when, in fact, there are no actual physical waves.



# Lifting set products to vector spaces

- Given two set universes  $U$  and  $W$ , the "composite" universe is their set product  $U \times W$ .
- Given two Hilbert spaces  $H_1$  and  $H_2$  with (orthonormal) bases  $\{|i\rangle\}$  and  $\{|j\rangle\}$ , we get the lifted vector space concept by applying the set concept to the basis sets and then generate the vector space concept.
- The set product of the bases  $\{|i\rangle\}$  and  $\{|j\rangle\}$  is the set of ordered pairs  $\{|i\rangle \otimes |j\rangle\}$  which generate the *tensor product*  $H_1 \otimes H_2$  (NB: not the direct product  $H_1 \times H_2$ ).

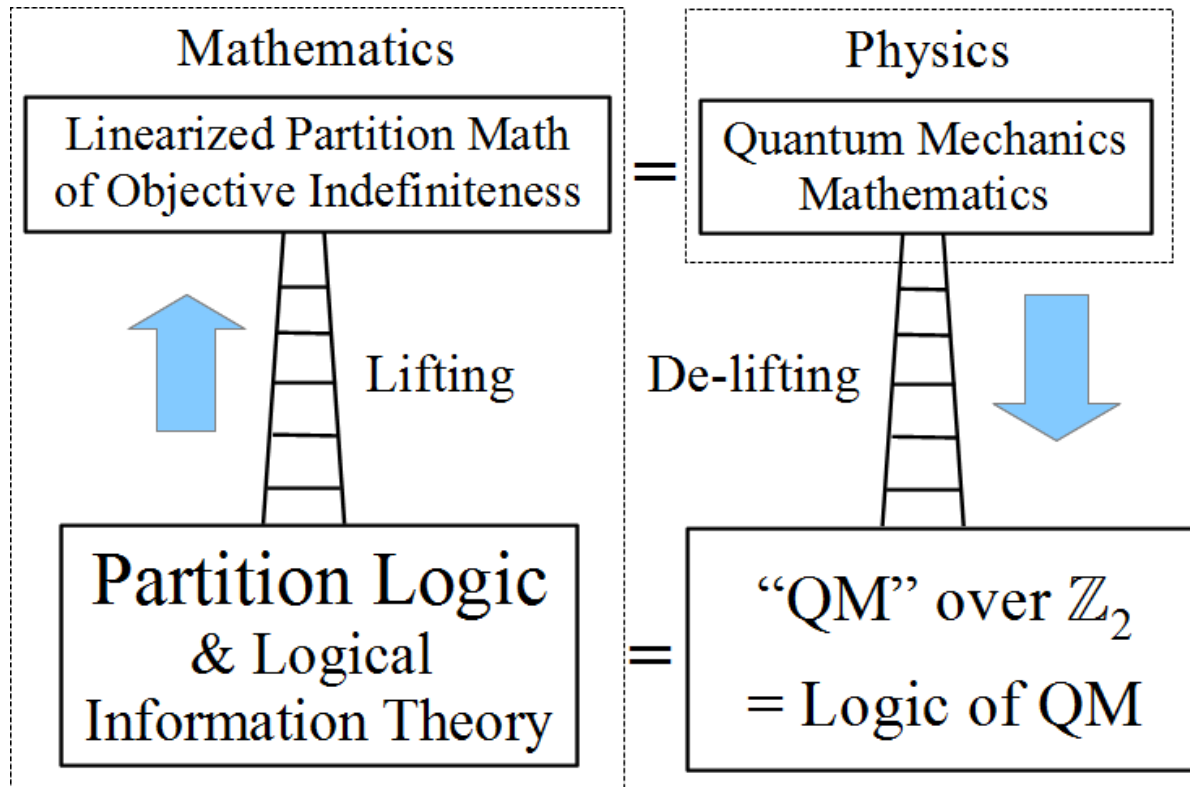
# "Deriving" QM math by lifting partition math



Thus by lifting partition math to vector spaces, we essentially get QM math: [abstract axioms based on Nielsen-Chuang book]

- Axiom 1: A system is represented by a unit vector in a complex vector space with inner product, i.e., Hilbert space. [lifting program]
- Axiom 2: Evolution of closed quantum system is described by a unitary transformation. [no-distinctions evolution]
- Axiom 3: A projective measurement for an observable (Hermitian operator)  $L = \sum_{\lambda} \lambda P_{\lambda}$  (spectral decomp.) on a pure state  $\rho$  has outcome  $\lambda$  with probability  $p_{\lambda} = \rho_{\lambda\lambda}$  giving mixed state  $\hat{\rho}$ . [density matrix treatment of measurement]
- Axiom 4: The state space of a composite system is the tensor product of the state spaces of component systems. [basis for tensor product = direct product of basis sets]
- No other interpretation of QM 'dreams' of essentially deriving axioms.

# The End



## **Objective Indefiniteness Interpretation of Quantum Mechanics**


as the back-story to the standard view:

superposition = complete description of quantum state.


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# Appendix 1: Group representations

- 
- Given group  $G$  indexing mappings  $\{R_g:U \rightarrow U\}_{g \in G}$ , what is required to make it set representation of the group  $G$ ?
  - Define  $u \sim u'$  if  $\exists g \in G$ , such that  $R_g(u) = u'$ .
    - $\exists R_1 = I_U = \text{identity}$  implies reflexivity of  $\sim$ ;
    - $\forall g \in G, \exists R_{g'} \text{ s.t. } R_g R_{g'} = R_1$  implies symmetry of  $\sim$ ;
    - $\forall g, g' \in G, \exists R_{gg'} \text{ s.t. } R_g R_{g'} = R_{gg'}$  implies transitivity of  $\sim$ .
  - $\{R_g:U \rightarrow U\}_{g \in G}$  is a set rep. of group implies  $\sim$  is an equivalence relation.
  - Set rep. of group = group action on  $U$  = 'dynamic' definition of an equivalence relation on  $U$ .
  - Group action defines indistinctions of a partition.
  - E.g., system after application of symmetry is indistinct from system before applying symmetry operation.

# Whence distinct eigen-alternatives?

- 
- Given the indistinctions defined by the set rep. or group action on  $U$ , what are all the distinct subsets that satisfy the indistinctions, i.e., that are not rendered indistinct by a group action?
  - Answer: An *invariant* subset  $S$  satisfies indistinctions, i.e.,  $R_g(S) \subseteq S \quad \forall g \in G$ , so maximally distinct subsets are the minimal invariant subsets = orbits = equivalence classes of  $\sim$ .
  - Orbits = distinct eigen-alternatives defined by set rep. or group action of group.

# Lifting from set reps. to vector space reps.



- Set rep. of  $G$  lifts to vector space rep.

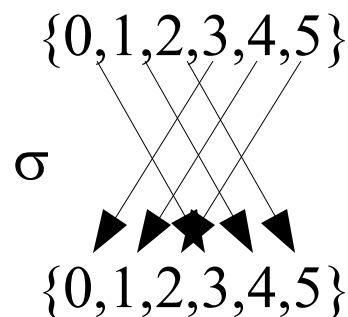
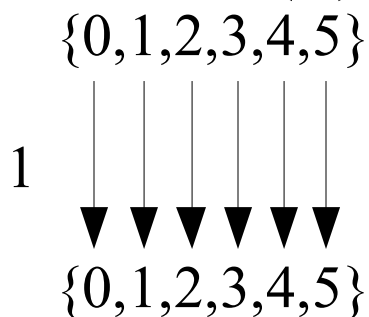
$$\{R_g: V \rightarrow V\}_{g \in G} \text{ where } R_1 = I_V \text{ and } R_g R_{g'} = R_{gg'}.$$

Set Representation lifts to $\rightarrow$	Vector space representation
Invariant subsets	Invariant subspaces
Orbits	Irreducible subspaces
Set partition of orbits	Vector space partition of irreducible subspaces
Representation restricted to orbit	Irreducible representation



# Example 1: Set Representation

- Set  $U = \{0,1,2,3,4,5\}$   $G = S_2 = \{1, \sigma\}$ , where  $R_\sigma(u) = u+3 \pmod{6}$ .




- 3 orbits:  $\{0,3\}$ ,  $\{1,4\}$ , and  $\{2,5\}$  which partition  $U$ .
- *Transitive* set rep. means only one orbit.
- Set rep. restricted to orbit, e.g.,  $\{0,3\}$ , is transitive.
- Symmetry-breaking = Move to subgroup, less indistinctions = more distinctions = refined partition of orbits.

# Commuting attributes and their lifts

- Set attribute  $f:U \rightarrow \mathbb{R}$  *commutes* with set rep. if the diagram commutes  $\forall g$ :
 
$$\begin{array}{ccc} U & \xrightarrow{R_g} & U \\ \downarrow f & & \downarrow f \\ \mathbb{R} & = & \mathbb{R} \end{array}$$
- Commuting real attribute lifts to Hermitian  $H$  operator commuting:  $\forall g, R_g H = H R_g$ .
- "Schur's Lemma" (on sets): commuting set attribute restricted to orbit is constant.
- Schur's Lemma: commuting operator restricted to irreducible subspace is constant operator.

## Example 2

- 
- $U = \{0, 1, \dots, 11\}$ ,  $G = S_2 = \{1, \sigma\}$  with set rep.  $R_\sigma(u) = u+6 \pmod{12}$ . Six orbits:  $\{0, 6\}$ ,  $\{1, 7\}$ ,  $\{2, 8\}$ ,  $\{3, 9\}$ ,  $\{4, 10\}$ , and  $\{5, 11\}$ .
  - Attribute  $f(n) = n \pmod{2}$  is a commuting attribute.
  - Partition:  $f^{-1}(0) = \{0, 2, 4, 6, 8, 10\}$ ,  $f^{-1}(1) = \{1, 3, 5, 7, 9, 11\}$
  - Attribute  $g(n) = n \pmod{3}$  is also commuting.
  - Partition:  $g^{-1}(0) = \{0, 3, 6, 9\}$ ,  $g^{-1}(1) = \{1, 4, 7, 10\}$ , and  $g^{-1}(2) = \{2, 5, 8, 11\}$ .
  - $f, g$  form a *Complete Set of Compatible Attributes* (CSCA):
    - $f^{-1}(0) \cap g^{-1}(0) = \{0, 6\} = |0, 0\rangle$ ,
    - $f^{-1}(0) \cap g^{-1}(1) = \{4, 10\} = |0, 1\rangle$ , etc.

# Lifting set reps. to vector space reps.

Lifting Program	Set group representations	Vector space group reps
Representation	Group $G$ represented by permutations $R_g:U \rightarrow U$	Group $G$ represented by invertible linear ops. $R_g:V \rightarrow V$
Min. invariants	Orbits	Irreducible subspaces
Partition	Set partition of orbits	Vector space partition of irreducible subspaces
Irreducible reps	Reps restricted to orbits	Reps restricted to irred. spaces
Commuting with representation	Attribute $f:U \rightarrow \mathbb{R}$ commuting with $R_g$ , i.e., $fR_g = f$ .	Operator $H$ commuting with $R_g$ , i.e., $HR_g = R_gH$
Invariants	Inverse-images $f^{-1}(r)$ for commuting $f$ are invariant.	Eigenspaces of commuting $H$ are invariant.
Schur's Lemma	Commuting $f$ restricted to orbit is constant function.	Commuting $H$ restricted to irred. subspace is constant op.

## Appendix 2: Indeterminacy principle in "QM" on sets

- In previous example of  $U = \{a, b, c\}$  and  $U' = \{a', b', c'\}$  where  $\{a'\} = \{a, b\}$ ,  $\{b'\} = \{b, c\}$ , and  $\{c'\} = \{a, b, c\}$ , let  $f$  be a real-valued attribute on  $U$  and  $g$  on  $U'$ .
- Don't have operators like  $L = \sum \lambda P_\lambda$  since only eigenvalues in  $\mathbb{Z}_2$  are 0,1, but we do have the projection operators like  $P_\lambda$ , namely  $f^{-1}(r) \cap ()$  and  $g^{-1}(s) \cap ()$ , so the commutativity properties are stated in terms of those projection operators.
- Let  $f = \chi_{\{b, c\}}$  and  $g = \chi_{\{a', b'\}}$ . The table shows they do not commute.

$U$	$U'$	$f \models \{b, c\} \cap ()$	$g \models \{a', b'\} \cap ()$	$g \models f \models$	$f \models g \models$
$\{a, b, c\}$	$\{c'\}$	$\{b, c\}$	$\emptyset$	$\{b, c\}$	$\emptyset$
$\{a, b\}$	$\{a'\}$	$\{b\}$	$\{a'\} = \{a, b\}$	$\{a, c\}$	$\{b\}$
$\{b, c\}$	$\{b'\}$	$\{b, c\}$	$\{b'\} = \{b, c\}$	$\{b, c\}$	$\{b, c\}$
$\{a, c\}$	$\{a', b'\}$	$\{c\}$	$\{a', b'\} = \{a, c\}$	$\{a, b\}$	$\{c\}$
$\{a\}$	$\{b', c'\}$	$\emptyset$	$\{b'\} = \{b, c\}$	$\emptyset$	$\{b, c\}$
$\{b\}$	$\{a', b', c'\}$	$\{b\}$	$\{a', b'\} = \{a, c\}$	$\{a, c\}$	$\{a, c\}$
$\{c\}$	$\{a', c'\}$	$\{c\}$	$\{a'\} = \{a, b\}$	$\{a, b\}$	$\{b\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Non-commutativity of the projections  $\{b, c\} \cap ()$  and  $\{a', b'\} \cap ()$ .

# Indeterminacy principle in "QM" on sets

- Define that two real-valued attributes  $f:U \rightarrow \mathbb{R}$  and  $g:U' \rightarrow \mathbb{R}$  "commute" iff their projectors  $f^{-1}(r) \cap ()$  and  $g^{-1}(s) \cap ()$  commute.

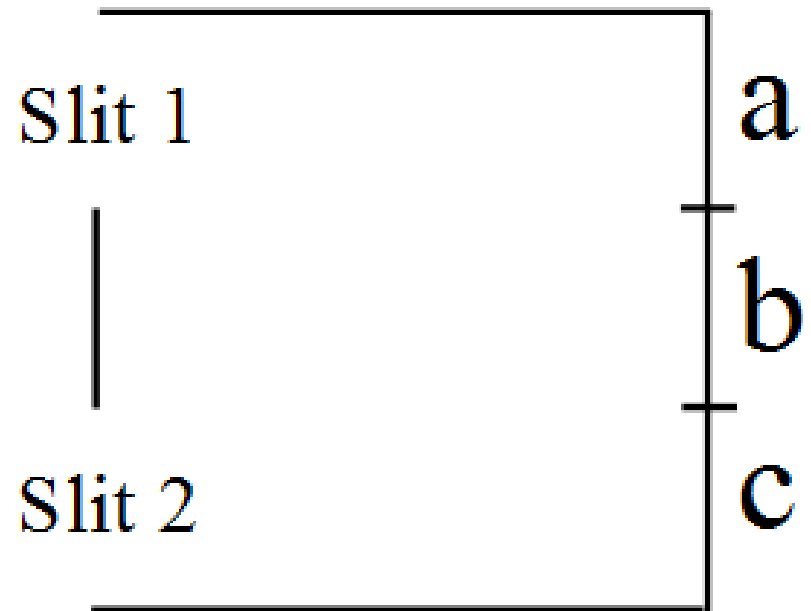
$U$	$U''$	$f \models \{a, b\} \cap ()$	$h \models \{a'', c''\} \cap ()$	$h \restriction f \restriction$	$f \restriction h \restriction$
$\{a, b, c\}$	$\{a'', c''\}$	$\{a, b\}$	$\{a'', c''\} = \{a, b, c\}$	$\{a, b\}$	$\{a, b\}$
$\{a, b\}$	$\{a''\}$	$\{a, b\}$	$\{a''\} = \{a, b\}$	$\{a, b\}$	$\{a, b\}$
$\{b, c\}$	$\{b'', c''\}$	$\{b\}$	$\{c''\} = \{c\}$	$\emptyset$	$\emptyset$
$\{a, c\}$	$\{a'', b'', c''\}$	$\{a\}$	$\{a'', c''\} = \{a, b, c\}$	$\{a, b\}$	$\{a, b\}$
$\{a\}$	$\{a'', b''\}$	$\{a\}$	$\{a''\} = \{a, b\}$	$\{a, b\}$	$\{a, b\}$
$\{b\}$	$\{b''\}$	$\{b\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{c\}$	$\{c''\}$	$\emptyset$	$\{c''\} = \{c\}$	$\emptyset$	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Commuting projection operators  $\{a, b\} \cap ()$  and  $\{a'', c''\} \cap ()$ .

- Theorem: Linear ops commute iff all their projectors commute iff there exists a basis of simultaneous eigenvectors.
- In this case, simult. basis is  $\{a, b\} = \{a''\}$ ,  $\{b\} = \{b''\}$ , and  $\{c\} = \{c''\}$ .
- This justifies previous defn:  $f$  and  $g$  compatible iff  $U = U'$ .

## Appendix 3: Two Slit Experiment in "QM" over $\mathbb{Z}_2$ : I

- Linear map  $A: \mathbb{Z}_2^{|U|} \rightarrow \mathbb{Z}_2^{|U|}$  that preserves distinctness is non-singular transformation (no inner product).
- For  $U = \{a, b, c\}$ , define A-dynamics by:  $\{a\} \rightarrow \{a, b\}$ ,  $\{b\} \rightarrow \{a, b, c\}$ , and  $\{c\} \rightarrow \{b, c\}$ .
- Let basis states  $\{a\}$ ,  $\{b\}$ , and  $\{c\}$  represent vertical "positions".
- Two slits on the left, and "particle" traverses box in 1 time period.
- "Particle" hits slits in indefinite state  $\{a, c\}$ .



# Two Slit Experiment in "QM" over $\mathbb{Z}_2$ : II



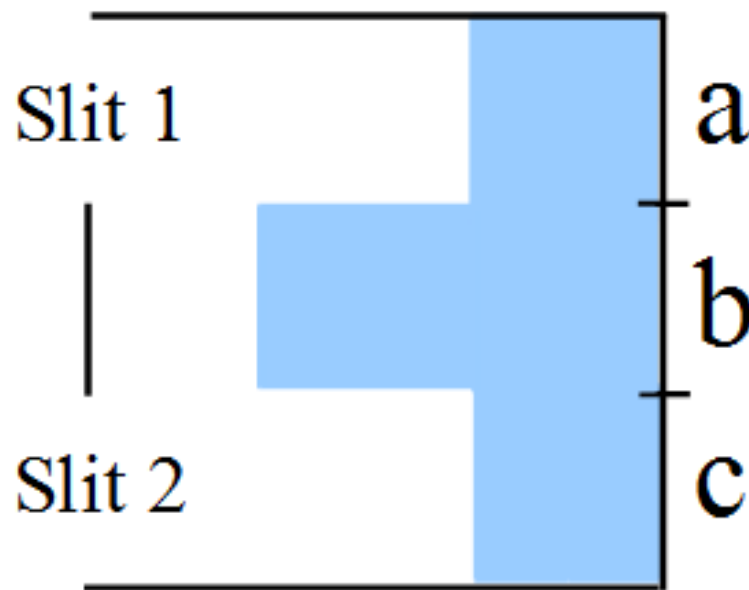
■ Case 1: "measurement," i.e., distinctions, at slits.

- $\Pr(\{a\}|\{a,c\}) = \frac{1}{2}$
- $\Pr(\{c\}|\{a,c\}) = \frac{1}{2}$ .

■ If  $\{a\}$ , then  $\{a\} \rightarrow \{a,b\}$ , and hits wall:  $\Pr(\{a\}|\{a,b\}) = \frac{1}{2} = \Pr(\{b\}|\{a,b\})$ .

■ If  $\{c\}$ , then  $\{c\} \rightarrow \{b,c\}$ , and hits wall:  $\Pr(\{b\}|\{b,c\}) = \frac{1}{2} = \Pr(\{c\}|\{b,c\})$ .

■ Thus at wall:  $\Pr(\{a\}) = \Pr(\{c\}) = \frac{1}{4}$  and  $\Pr(\{b\}) = \frac{1}{2}$ .





# Two Slit Experiment in "QM" over $\mathbb{Z}_2$ : III



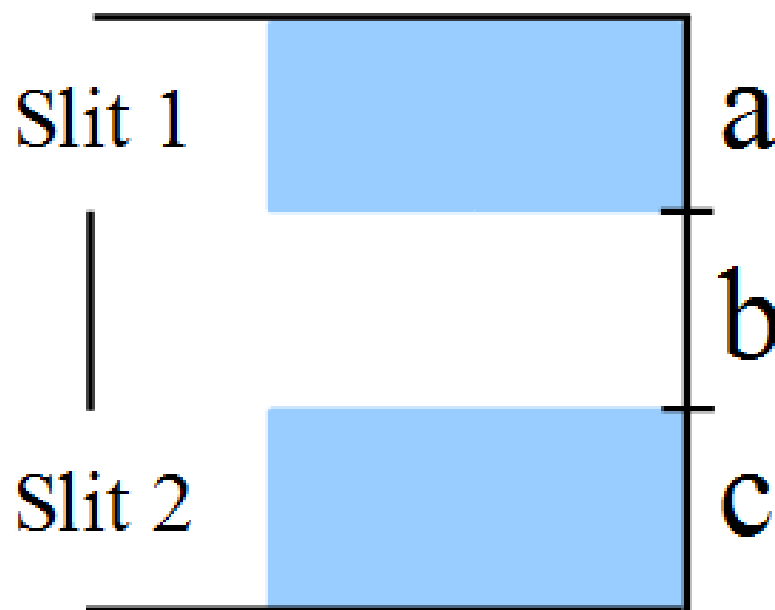
■ Case 2: no "measurement," i.e., no distinctions, at slits.

■  $\{a, c\}$  evolves linearly:

- $\{a\} \rightarrow \{a, b\}$  and
- $\{c\} \rightarrow \{b, c\}$  so that:
- $\{a\} + \{c\} = \{a, c\} \rightarrow \{a, b\} + \{b, c\} = \{a, c\}.$

■ At the wall,  $\Pr(\{a\}|\{a, c\}) = \frac{1}{2} :$   
 $\Pr(\{c\}|\{a, c\}).$

■ "Interference" cancels  $\{b\}$  in:  
 $\{a, c\} \rightarrow \{a, b\} + \{b, c\} = \{a, c\}.$



# Appendix 4: Entanglement in "QM" over $\mathbb{Z}_2$



- Basis principle: direct product  $X \times Y$  lifts to the tensor product  $V \otimes W$  of vector spaces.
- Subsets of  $X$ ,  $Y$ , and  $X \times Y$  correlate (via delifting-lifting) to vectors in  $V$ ,  $W$ , and  $V \otimes W$ .
- For  $S_X \subseteq X$  and  $S_Y \subseteq Y$ ,  $S_X \times S_Y \subseteq X \times Y$  is "separated" correlates with  $v \in V$  and  $w \in W$  giving separated  $v \otimes w \in V \otimes W$ .
- "Entangled" = Not "separated" subset  $S \subseteq X \times Y$ .
- Joint prob. dist.  $\Pr(x,y)$  on  $X \times Y$  is *correlated* if  $\Pr(x,y) \neq \Pr(x)\Pr(y)$  for marginals  $\Pr(x)$  and  $\Pr(y)$ .
- Theorem:  $S \subseteq X \times Y$  is "entangled" iff equiprobable distribution on  $S$  is correlated.

# Appendix 4: Bell inequality in "QM" over $\mathbb{Z}_2$ : I



- Consider  $\mathbb{Z}_2^2$  with three incompatible bases  $U=\{a,b\}$ ,  $U'=\{a',b'\}$ , and  $U''=\{a'',b''\}$  related as in the ket table.
- Given one of the kets as initial state, measurements in each basis have these probs.

kets	$U$ -basis	$U'$ -basis	$U''$ -basis
$ 1\rangle$	$\{a, b\}$	$\{a'\}$	$\{a''\}$
$ 2\rangle$	$\{b\}$	$\{b'\}$	$\{a'', b''\}$
$ 3\rangle$	$\{a\}$	$\{a', b'\}$	$\{b''\}$
$ 4\rangle$	$\emptyset$	$\emptyset$	$\emptyset$

Ket table for  $\wp(U) \cong \wp(U') \cong \wp(U'') \cong \mathbb{Z}_2^2$ .

Given state \ Outcome of test	$a$	$b$	$a'$	$b'$	$a''$	$b''$
$\{a, b\} = \{a'\} = \{a''\}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	1	0
$\{b\} = \{b'\} = \{a'', b''\}$	0	1	0	1	$\frac{1}{2}$	$\frac{1}{2}$
$\{a\} = \{a', b'\} = \{b''\}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	1

State-outcome table.

# Bell inequality in "QM" over $\mathbb{Z}_2$ : II

- Now form  $U \times U$  and compute the kets.
- Since  $\{a\} = \{a', b'\} = \{b''\}$  and  $\{b\} = \{b'\} = \{a'', b''\}$ ,  

$$\{(a, b)\} = \{a\} \times \{b\} = \{a', b'\} \times \{b'\} = \{(a', b'), (b', b')\}$$

$$= \{b''\} \times \{a'', b''\} = \{(b'', a''), (b'', b'')\}.$$
- Ket table has 16 rows of these relations but we need the one for an "entangled Bell state":  

$$\{(a, a), (b, b)\} = \{(a', a'), (a', b'), (b', a'), (b', b')\} + \{(b', b')\}$$

$$= \{(a', a'), (a', b'), (b', a')\} = \{(a'', a''), (a'', b''), (b'', a'')\}.$$

# Bell inequality in "QM" over $\mathbb{Z}_2$ : III

- Define prob. dist.  $\Pr(x,y,z)$  for probability:
  - getting  $x$  in  $U$ -measurement on left-hand system, &
  - if instead, getting  $y$  in  $U'$ -meas. on left-hand system, &
  - if instead, getting  $z$  in  $U''$ -meas. on left-hand system.
- For instance,  $\Pr(a,a',a'') = (1/2)(2/3)(2/3) = 2/9$ .
- Then consider the marginals:
  - $\Pr(a,a') = \Pr(a,a',a'') + \Pr(a,a',b'')^*$
  - $\Pr(b',b'') = \Pr(a,b',b'')^* + \Pr(b,b',b'')$
  - $\Pr(a,b'') = \Pr(a,a',b'')^* + \Pr(a,b',b'')^*$ .
- Since probs with asterisks in last row occur in other rows and since all probs are non-negative:

$$\Pr(a,a') + \Pr(b',b'') \geq \Pr(a,b'')$$

Bell Inequality

# Bell inequality in "QM" over $\mathbb{Z}_2$ : IV

- Consider *independence assumption*: outcome of test on right-hand system independent of test on left-hand system.
- For given initial state,
 
$$\{(a,a),(b,b)\} = \{(a',a'),(a',b'),(b',a')\} = \{(a'',a''),(a'',b''),(b'',a'')\},$$
 outcomes of initial tests on LH and RH systems have same probabilities.
- Hence prob. distributions  $\Pr(x,y)$ ,  $\Pr(y,z)$ , and  $\Pr(x,z)$  would be the same (under independence) if second variable always referred to test on *right-hand* system.
- With same probs., Bell inequality still holds.

# Bell inequality in "QM" over $\mathbb{Z}_2$ : V

- Given state:  $\{(a,a),(b,b)\} = \{(a',a'),(a',b'),(b',a')\} = \{(a'',a''),(a'',b''),(b'',a'')\}$
- To see if independence assumption is compatible with "QM" on sets, we compute the probs.
  - $\Pr(a,a')$  gets  $\{a\}$  with prob.  $\frac{1}{2}$  but then state of RH system is  $\{a\}$  so prob. of  $\{a'\}$  is  $\frac{1}{2}$  (see state-outcome table) so  $\Pr(a,a') = \frac{1}{4}$ .
  - $\Pr(b',b'')$  gets  $\{b'\}$  with prob.  $\frac{1}{3}$  but then state of RH system is  $\{a'\}$  and prob. of  $\{b''\}$  is 0, so  $\Pr(b',b'') = 0$ .
  - $\Pr(a,b'')$  gets  $\{a\}$  with prob.  $\frac{1}{2}$  but then state of RH system is  $\{a\}$  so prob. of  $\{b''\}$  is 1, so  $\Pr(a,b'') = \frac{1}{2}$ .
- Plugging into Bell inequality:  $\Pr(a,a') + \Pr(b',b'') \geq \Pr(a,b'')$  gives:  $\frac{1}{4} + 0 \geq \frac{1}{2}$  which is false!
- Hence independence fails & "QM" on sets is "nonlocal."

## Appendix 5: Measurement problem in "QM" over $\mathbb{Z}_2$ : I

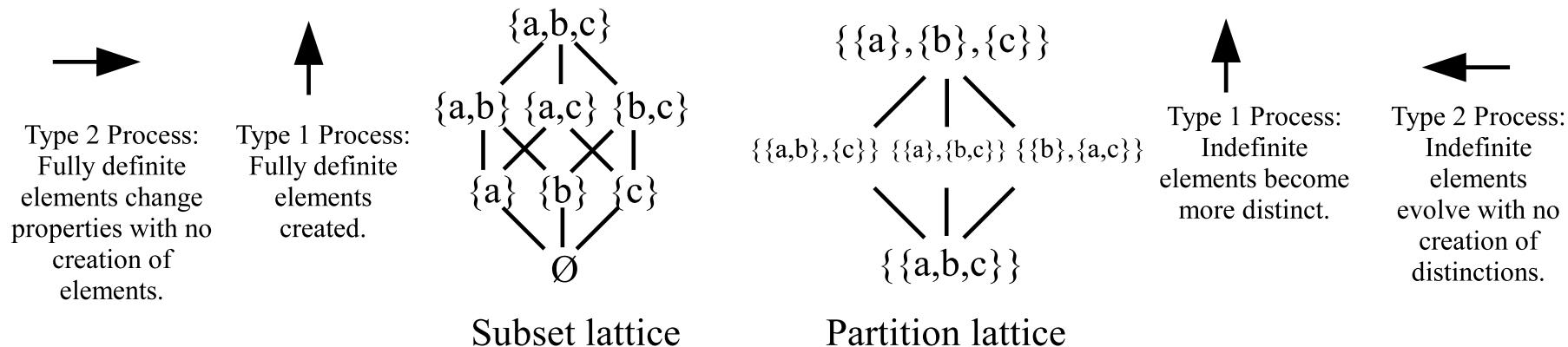
- von Neumann's terminology:
  - Type 1 process: quantum jump as in measurement;
  - Type 2 process: unitary transformation.
- Measurement problem is accounting for type 1 processes.
- In "QM" over  $\mathbb{Z}_2$ , type 2 process is *non-singular* transformation  $A$ , i.e., one that preserves "brackets" taking into account the change of basis where  $Au = u'$  so  $A(U) = U'$ :

$$\langle S|_U T \rangle = |S \cap T|_U = |S' \cap T'|_{U'} = \langle S'|_{U'} T' \rangle.$$



# Measurement problem in "QM" over $\mathbb{Z}_2$ : II

- In "QM" over  $\mathbb{Z}_2$ :
  - Type 2 process = distinction-preserving;
  - Type 1 process = distinction-making.
- In tale of two lattices, elements  $\sim$  distinctions so element-creating is the "classical" version of type 1 distinction-creating processes.
- And indeed, the process of creating elements certainly cannot be described by the type 2 evolution of classical mechanics.



# Measurement problem in "QM" over $\mathbb{Z}_2$ : III

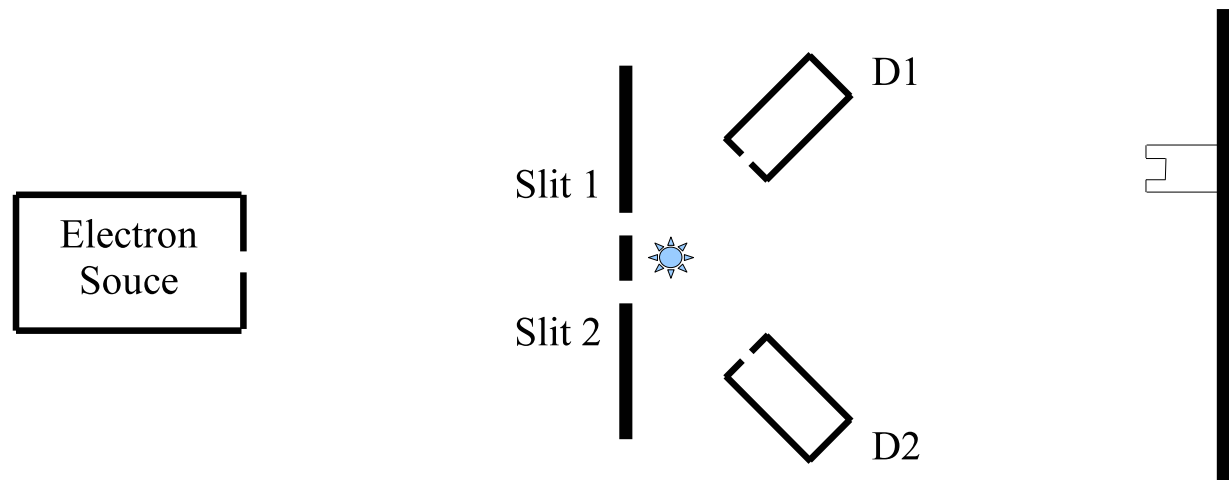


- Similarly, in "QM" over  $\mathbb{Z}_2$ , type 1 distinction-creating processes cannot be explained by unitary (Schrödinger equation) type 2 processes.
- Principal distinction-making operation is the *join-operation* where an attribute  $f:U \rightarrow \mathbb{R}$  slices up a subset  $S$  into blocks  $\{f^{-1}(r_i) \cap S \mid r_i \in \text{Im}(f)\}$  in a  $f$ -measurement.
- Probability:  $\Pr(r_i|S) = |f^{-1}(r_i) \cap S|/|S|$ .
- Lifting to QM: how are physical distinctions made?

# Measurement problem in "QM" over $\mathbb{Z}_2$ : IV

- Old problem of finding physically distinguishing events to perform a measurement.
- Two-slit experiment: why are  $|\text{slit 1}\rangle$  and  $|\text{slit 2}\rangle$  superposable but  $|\text{detector 1}\rangle$  and  $|\text{detector 2}\rangle$  not?
- "QM" over  $\mathbb{Z}_2$  gives no physics answer; only conceptual answer of non-superposable distinguishing events.

"You must never add amplitudes for different and distinct final states. ...You do add the amplitudes for the different indistinguishable alternatives inside the experiment." Feynman



# Measurement problem in "QM" over $\mathbb{Z}_2$ : V



- Modeling measurement:  $Q = \{a,b\}$ ;  $M = \{0,1,2\}$ .
- Assumption: indicator states  $M$  not superposable.
- Composite system:  $Q \times M \cong \mathbb{Z}_2^6$ ;
- Initial state:  $\{(a,0)\} + \{(b,0)\} = \{(a,0),(b,0)\}$ .
- Apply type 2 non-singular transform defined by:
 

$(a,0) \rightarrow (a,1)$
$(b,0) \rightarrow (b,2)$
$(a,1) \rightarrow (a,0)$
$(b,1) \rightarrow (b,1)$
$(a,2) \rightarrow (a,2)$
$(b,2) \rightarrow (b,0)$
- Result is:  $\{(a,1),(b,2)\}$  = superposition of indicator states (like Schrödinger's cat).
- Then distinctions are made by join-action of partition
 
$$0_Q \times 1_M = \{ \{(a,0),(b,0)\}, \{(a,1),(b,1)\}, \{(a,2),(b,2)\} \}.$$
- Result is mixed state:  $\{(a,1)\}, \{(b,2)\}$  with half-half prob.

# Appendix 6: Quantum Mechanics over $\mathbb{Z}_2$

QM over  $\mathbb{Z}_2$  as the new quantum logic

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April 2013

# New Quantum Logic = "QM" over $\mathbb{Z}_2$

- Logic of  $X$  = bare-bones essence of  $X$ .
- Proposed new Quantum Logic = QM math
  - Distilled down to 0, 1 aspects, i.e., from  $\mathbb{C}$  to  $\mathbb{Z}_2$  as base field,
  - Without metrical aspects in "quantum states",
  - No physical assumptions.
- Take powerset  $\wp(U)$  as vector space  $\mathbb{Z}_2^n$  over  $\mathbb{Z}_2$  where  $|U| = n$ .
- Vector addition = symmetric difference, i.e.,  
$$S + T = S \cup T - S \cap T.$$
- Each subset, e.g.,  $\{a, b\} \subseteq \{a, b, c, d\}$ , is vector sum:  
 $\{a\} + \{b\}$  in  $\mathbb{Z}_2^n$ .

# Kets in a vector space over $\mathbb{Z}_2$

- Consider  $U = \{a, b, c\}$  and  $U' = \{a', b', c'\}$  where:
  - $\{a'\} = \{a, b\} = \{a\} + \{b\},$
  - $\{b'\} = \{b, c\} = \{b\} + \{c\},$
  - $\{c'\} = \{a, b, c\} = \{a\} + \{b\} + \{c\}.$
  - Therefore, say,  
 $\{b', c'\} = \{b'\} + \{c'\} = \{b, c\} + \{a, b, c\} = \{a\}.$

U	{a,b,c}	{a,b}	{b,c}	{a,c}	{a}	{b}	{c}	$\emptyset$
U'	{c'}	{a'}	{b'}	{a',b'}	{b',c'}	{a',b',c'}	{a',c'}	$\emptyset$

Ket table for  $\mathbb{Z}_2^3$ : Columns = kets

- Each column gives a ket (abstract vector) expressed in different bases.





# Spectral decomposition of attribute $f$ : I

- Attribute  $f : U \rightarrow \mathbb{R}$  restricted to small enough subset  $S \subseteq U$  has a constant value  $r$  so we formally write:

$$f \upharpoonright S = rS$$

Set version of eigenvalue equation:  $Lv = \lambda v$ .

- *Eigenspaces* are  $\wp(f^{-1}(r))$  for eigenvalue  $r$  and *projections* to eigenspaces are:

$$f^{-1}(r) \cap () : \wp(U) \rightarrow \wp(U)$$

Set version of projections  $P_\lambda$  to eigenspace  $E_\lambda$  for  $\lambda$ .

- Using this formal convention, we have:

# Spectral decomposition of attribute $f$ : II

$$f = \sum_r r [f^{-1}(r) \cap ()].$$

Spectral decomposition of  $f : U \rightarrow \mathbb{R}$

Set version of:  $L = \sum_{\lambda} \lambda P_{\lambda}$ .

- Vector space (over  $\mathbb{Z}_2$ ) *partition* of eigenspaces:

$$\wp(f^{-1}(r)) \oplus \dots \oplus \wp(f^{-1}(r')) = \mathbb{Z}_2^{|U|}$$

Set version of  $\sum_{\lambda} \oplus E_{\lambda} = V$ .

- *Completeness* of eigenspace projections:

$$\sum_r [f^{-1}(r) \cap ()] = I : \wp(U) \rightarrow \wp(U)$$

Set version of  $\sum_{\lambda} P_{\lambda} = I : V \rightarrow V$ .

# Spectral decomposition of attribute $f$ : III

- *Orthogonality* of eigenspace projections: (note orthogonality of projections well-defined without inner product)

For  $r \neq r'$ ,  $[f^{-1}(r) \cap ()] [f^{-1}(r') \cap ()] = \emptyset \cap () : \wp(U) \rightarrow \wp(U)$   
 Set version of: for  $\lambda \neq \lambda'$ ,  $P_\lambda P_{\lambda'} = 0 : V \rightarrow V$ .

- Summary:

"QM" over $\mathbb{Z}_2$	QM over $\mathbb{C}$
$S \cap () : \wp(U) \rightarrow \wp(U)$	$P : V \rightarrow V$
$f \upharpoonright () = \sum_r r (f^{-1}(r) \cap ())$	$L = \sum_\lambda \lambda P_\lambda$
$\sum_r f^{-1}(r) \cap () = I : \wp(U) \rightarrow \wp(U)$	$\sum_\lambda P_\lambda = I$
$r \neq r', [f^{-1}(r) \cap ()] [f^{-1}(r') \cap ()] = \emptyset \cap ()$	$\lambda \neq \lambda', P_\lambda P_{\lambda'} = 0$

# Brackets in QM over $\mathbb{Z}_2$ : I

- No inner products in vector spaces over finite fields, but the set version of Dirac's bra-kets can still be defined at the cost of basis-dependent bras.
- Kets  $|T\rangle$  are basis-free, but bras  $\langle S|_U$  are basis-dependent as indicated by subscript: For  $S, T \subseteq U$ ,

$$\begin{aligned}\langle S|_U T\rangle &= |S \cap T| \\ &= \text{cardinality of overlap of } S \text{ and } T.\end{aligned}$$

- Basis Principle for lifting: Apply set concept to set of basis vectors and generate the corresponding vector space concept.

# Brackets in QM over $\mathbb{Z}_2$ : II

- Basis Principle for delifting: Define vectors  $\psi_S = \sum_{v_i \in S} |v_i\rangle$  corresponding to subsets  $S \subseteq \{|v_i\rangle\}$  of a basis set  $\{|v_i\rangle\}$ , and then apply vector space concept to those vectors to suggest set concept.
- Given two subsets  $S, T \subseteq \{|v_i\rangle\}$ ,  $\psi_S = \sum_{v_i \in S} |v_i\rangle$  and  $\psi_T = \sum_{v_i \in T} |v_i\rangle$ , then the *overlap* between the state-vectors  $\psi_S$  and  $\psi_T$  is the inner product  $\langle \psi_S | \psi_T \rangle = |S \cap T|$ .
- This motivates and confirms the definition: for  $S, T \subseteq U$

$$\langle S |_U T \rangle = |S \cap T|$$

Brackets in QM over  $\mathbb{Z}_2$ .

- Thus  $\langle S |_U : \wp(U) \rightarrow \mathbb{R}$  is the basis-dependent set version of the basis-independent bra  $\langle v | : V \rightarrow \mathbb{C}$ .

# Brackets in QM over $\mathbb{Z}_2$ : III

- The idea of a real-valued basis-dependent function on vectors in vector spaces over  $\mathbb{Z}_2$  is standard in coding theory. Given two ordered  $n$ -tuples of 0, 1s,  $S$  and  $T$ , then *Hamming distance function* is  $d(S, T) = |S + T|$  where the addition is the symmetric difference so it is the number of places where the binary strings differ.
- For  $u \in U$ , the *ket-bra*  $|\{u\}\rangle \langle \{u\}|_U = \{u\} \cap () : \wp(U) \rightarrow \wp(U)$  is defined as that projection operator.
- *Completeness* of ket-bra sum:

$$\sum_{u \in U} |\{u\}\rangle \langle \{u\}|_U = I : \wp(U) \rightarrow \wp(U)$$

Set version of:  $\sum_i |v_i\rangle \langle v_i| = I : V \rightarrow V.$

# Brackets in QM over $\mathbb{Z}_2$ : IV

- *Resolution of unity* by ket-bra sums:

$$\langle S|_U T \rangle = \sum_u \langle S|_U \{u\} \rangle \langle \{u\} |_U T \rangle = |S \cap T| \text{ for } S, T \subseteq U$$

Set version of:  $\langle \psi | \varphi \rangle = \sum_i \langle \psi | v_i \rangle \langle v_i | \varphi \rangle.$

- Then  $\langle \{u\} |_U S \rangle$  is the *amplitude* of  $\{u\}$  in  $S$ , i.e.,

$$\langle \{u\} |_U S \rangle = \chi_S(u) = \begin{cases} 1 & \text{if } u \in S \\ 0 & \text{if } u \notin S \end{cases}.$$

# Example: Resolving a ket

- Also a ket  $|S\rangle$  can be resolved in the  $U$ -basis: for  $S \subseteq U$ ,  
 $|S\rangle = \sum_{u \in U} \langle \{u\} |_U S \rangle |\{u\}\rangle$  where  $\langle \{u\} |_U S \rangle =$  *amplitude of  $\{u\}$  in  $S$*   
Set version of:  $|\psi\rangle = \sum_i \langle v_i | \psi \rangle |v_i\rangle$  where  $\langle v_i | \psi \rangle =$  *amplitude of  $v_i$  in  $\psi$* .
- Using the previous example,  $|\{a', b'\}\rangle$  can be resolved in the  $U$ -basis.

$$\begin{aligned} & |\{a', b'\}\rangle \\ &= \langle \{a\} |_U \{a', b'\} \rangle |\{a\}\rangle \\ &+ \langle \{b\} |_U \{a', b'\} \rangle |\{b\}\rangle \\ &+ \langle \{c\} |_U \{a', b'\} \rangle |\{c\}\rangle \\ &\quad \text{(then using } \{a', b'\} = \{a, c\} \text{)} \\ &= |\{a\} \cap \{a, c\}| |\{a\}\rangle + |\{b\} \cap \{a, c\}| |\{b\}\rangle + |\{c\} \cap \{a, c\}| |\{c\}\rangle \\ &= |\{a\}\rangle + |\{c\}\rangle. \end{aligned}$$



# Summary so far

QM over $\mathbb{Z}_2$	QM over $\mathbb{C}$
$S \cap () : \wp(U) \rightarrow \wp(U)$	$P : V \rightarrow V$
$f \upharpoonright () = \sum_r r (f^{-1}(r) \cap ())$	$L = \sum_\lambda \lambda P_\lambda$
$\sum_r f^{-1}(r) \cap () = I : \wp(U) \rightarrow \wp(U)$	$\sum_\lambda P_\lambda = I$
$[f^{-1}(r) \cap ()] [f^{-1}(r') \cap ()] = \emptyset \cap ()$	$P_\lambda P_{\lambda'} = 0$
$\langle S _U T \rangle =  S \cap T  = \text{overlap } S, T \subseteq U$	$\langle \psi   \varphi \rangle = \text{overlap } \psi, \varphi$
$\sum_{u \in U}  \{u\}\rangle \langle \{u\} _U = I$	$\sum_i  v_i\rangle \langle v_i  = I$
$\langle S _U T \rangle = \sum_u \langle S _U \{u\} \rangle \langle \{u\} _U T \rangle$	$\langle \psi   \varphi \rangle = \sum_i \langle \psi   v_i \rangle \langle v_i   \varphi \rangle$

# Magnitude (or norm): I

- Notation conflict: In QM and complex numbers, the *magnitude* or *absolute value*  $|v| = \sqrt{\langle v|v \rangle}$  is indicated with the single bars  $|v|$ . But in QM over  $\mathbb{Z}_2$ , the single bars are the standard notation for the cardinality  $|S|$  so we use the alternative norm-notation  $\|S\|_U$  for the "magnitude" in the set case.
- Norm or magnitude of a vector in  $\mathbb{Z}_2^n$  is: for  $S \subseteq U$ ,

$$\|S\|_U = \sqrt{\langle S|_U S \rangle} = \sqrt{|S|}$$

Set version of:  $|\psi| = \sqrt{\langle \psi|\psi \rangle}$ .

- Resolution of unity applied to magnitude squared:

# Magnitude (or norm): II

$$\|S\|_U^2 = \langle S|_U S \rangle = \sum_{u \in U} \langle S|_U \{u\} \rangle \langle \{u\} |_U S \rangle = |S|$$

Set version of:

$$|\psi|^2 = \langle \psi | \psi \rangle = \sum_i \langle \psi | v_i \rangle \langle v_i | \psi \rangle = \sum_i \langle v_i | \psi \rangle^* \langle v_i | \psi \rangle$$

- Normalization to get Laplacian probabilities: for  $S \neq \emptyset$

$$\sum_{u \in U} \frac{\langle S|_U \{u\} \rangle \langle \{u\} |_U S \rangle}{\|S\|_U^2} = \sum_{u \in S} \frac{1}{|S|} = 1$$

$$\text{Set version of: } \sum_i \frac{\langle v_i | \psi \rangle^* \langle v_i | \psi \rangle}{\langle \psi | \psi \rangle} = 1 \text{ for } \psi \neq 0.$$

- Note how Laplacian equi-probabilities are derived from the squares of the basis-coefficient values  $\langle S|_U \{u\} \rangle \langle \{u\} |_U S \rangle$  constituting the state  $S \subseteq U$ , just as the QM probabilities are derived from the absolute squares of the basis-coefficient values:  $\langle v_i | \psi \rangle^* \langle v_i | \psi \rangle$  (after normalization in both cases).

# Magnitude (or norm): III

Laplacian probability of  $u$  given  $S$ :  $\Pr(u|S) = \frac{\langle S|u\{u\}\rangle\langle\{u\}|uS\rangle}{\|S\|_u^2}$

Set version of QM probability: abs. sq. amplitude =  $\frac{\langle v_i|\psi\rangle^*\langle v_i|\psi\rangle}{\langle\psi|\psi\rangle}$ .

- Much ado is made of quantum probabilities being the absolute squares of the amplitudes constituting the state  $\psi$ , but now we see that the same is true in ordinary logical finite probability theory once formulated using the concepts of QM over  $\mathbb{Z}_2$ , i.e.,

$$\Pr(u|S) = \frac{\langle\{u\}|uS\rangle^2}{\langle S|uS\rangle} = \begin{cases} 1/|S| & \text{if } u \in S \\ 0 & \text{if } u \notin S \end{cases}.$$

# Attributes (RVs) and probabilities: I

- A real-valued attribute  $f : U \rightarrow \mathbb{R}$  is a random variable on the Laplacian (equiprobable) outcome space  $U$ .
- Given  $S \subseteq U, S \neq \emptyset$ :

$$\|S\|_U^2 = \langle S|_U S \rangle = \sum_r \|f^{-1}(r) \cap S\|_U^2 = \sum_r |f^{-1}(r) \cap S| = |S|$$

Set version of:  $|\psi|^2 = \langle \psi | \psi \rangle = \sum_\lambda |P_\lambda(\psi)|^2$ .

- Normalized sum: for  $S \neq \emptyset$ ,

$$\sum_r \frac{\|f^{-1}(r) \cap S\|_U^2}{\|S\|_U^2} = \sum_r \frac{|f^{-1}(r) \cap S|}{|S|} = 1$$

Set version of:  $\sum_\lambda \frac{|P_\lambda(\psi)|^2}{|\psi|^2} = 1$  for  $\psi \neq 0$ .

# Attributes (RVs) and probabilities: II

- Given the set  $S$ , the probability that the random variable  $f : U \rightarrow \mathbb{R}$  has the value  $r$  is:

$$\Pr(r|S) = \frac{\|f^{-1}(r) \cap S\|_U^2}{\|S\|_U^2} = \frac{|f^{-1}(r) \cap S|}{|S|}$$

$$\text{Set version of Born Rule: } \Pr(\lambda | \psi) = \frac{|P_\lambda(\psi)|^2}{|\psi|^2}.$$

- Nota Bene* that the set-version of quantum mechanic's Born Rule is not some mysterious "quantum probability" but is the perfectly ordinary Laplace-Boole logical finite probability of a random variable  $f$  taking a certain value  $r$  given a subset  $S$ .

# Example: Probability of two dice giving a 7

- Outcome space  $U = \{1, \dots, 6\} \times \{1, \dots, 6\}$ .
- Random variable  $f : U \rightarrow \mathbb{R}$  where  $f(d_1, d_2) = d_1 + d_2$ .
- $f^{-1}(7) = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$ .
- Take  $S = U$  so  $|f^{-1}(7) \cap U| = 6$  and  $|U| = 36$ .
- Laplacian probability of getting a 7 is:

$$\Pr(7|U) = \frac{\|f^{-1}(7) \cap U\|_U^2}{\|U\|_U^2} = \frac{|f^{-1}(7) \cap U|}{|U|} = \frac{6}{36} = \frac{1}{6}.$$

# Average value of an attribute-observable

- Given an attribute-observable  $f : U \rightarrow \mathbb{R}$ , the spectral decomposition is  $f \upharpoonright () = \sum_r r [f^{-1}(r) \cap ()]$  so that:

$$\langle S | f \upharpoonright () | S \rangle = \langle S | \sum_r r f^{-1}(r) \cap () | S \rangle = \sum_r r \langle S | f^{-1}(r) \cap S \rangle = \sum_r r |f^{-1}(r) \cap S|.$$

- Then normalizing by  $\langle S | U S \rangle$  gives:

$$\langle f \rangle_S = \frac{\langle S | f \upharpoonright () | S \rangle}{\langle S | U S \rangle} = \sum_r r \frac{|f^{-1}(r) \cap S|}{|S|} = \sum_r r \Pr(r|S) = \text{ave. of } f \text{ on } S$$

Set version of:  $\langle L \rangle_\psi = \frac{\langle \psi | L | \psi \rangle}{\langle \psi | \psi \rangle} = \text{average value of observable } L \text{ in state } \psi.$



# Remaining summary table

QM over $\mathbb{Z}_2$	QM over $\mathbb{C}$
$\ S\ _U = \sqrt{\langle S _U S \rangle} = \sqrt{ S }$ for $S \subseteq U$	$ \psi  = \sqrt{\langle \psi   \psi \rangle}$
$\ S\ _U^2 = \sum_{u \in U} \langle \{u\}  _U S \rangle^2 =  S $	$ \psi ^2 = \sum_i \langle v_i   \psi \rangle^* \langle v_i   \psi \rangle$
$\sum_{u \in U} \frac{\langle \{u\}  _U S \rangle^2}{\ S\ _U^2} = \sum_{u \in S} \frac{1}{ S } = 1$	$\sum_i \frac{\langle v_i   \psi \rangle^* \langle v_i   \psi \rangle}{ \psi ^2} = 1$
$\ S\ _U^2 = \sum_r  f^{-1}(r) \cap S  =  S $	$ \psi ^2 = \sum_\lambda  P_\lambda(\psi) ^2$
$\sum_r \frac{\ f^{-1}(r) \cap S\ _U^2}{\ S\ _U^2} = \sum_r \frac{ f^{-1}(r) \cap S }{ S } = 1$	$\sum_\lambda \frac{ P_\lambda(\psi) ^2}{ \psi ^2} = 1$
$\Pr(r S) = \frac{\ f^{-1}(r) \cap S\ _U^2}{\ S\ _U^2} = \frac{ f^{-1}(r) \cap S }{ S }$	$\Pr(\lambda \psi) = \frac{ P_\lambda(\psi) ^2}{ \psi ^2}$
$\langle f \rangle_S = \frac{\langle S  _U f^\dagger()   S \rangle}{\langle S  _U S \rangle}$	$\langle L \rangle_\psi = \frac{\langle \psi   L   \psi \rangle}{\langle \psi   \psi \rangle}$

$\mathbb{Z}_2 \Rightarrow \mathbb{C}$  takes quantum logic to quantum mechanics

# Objective indefiniteness in QM over 2: I

- Collecting elements  $a, b \in U = \{a, b, c\}$  together in the subset  $\{a, b\}$  is the set version of superposition of basis vectors  $v_1 + v_2$ .
- But the subset  $\{a, b\}$  is *not* interpreted as a subset of definite elements  $a, b$  but as a single indefinite element that is indefinite between  $\{a\}$  and  $\{b\}$ , just as  $\frac{1}{\sqrt{2}}(v_1 + v_2)$  is a quantum state that is objectively indefinite between the two eigenstates  $v_1$  and  $v_2$ .
- Probabilities also have an objective indefiniteness interpretation.
  - $S$  is a single indefinite element, and the probability that the indefinite element  $S$  will reduce to the definite element  $\{u\}$  in a nondegenerate measurement using the  $U$ -basis is:

# Objective indefiniteness in QM over $\mathbb{Z}_2$ : II

$$\Pr(u|S) = \frac{\langle S|_U \{u\} \rangle \langle \{u\}|_U S \rangle}{\langle S|_U S \rangle} = \frac{1}{|S|}.$$

- Similarly in QM over  $\mathbb{C}$ ,
  - $\psi$  is an objectively indefinite state, and the probability that the indefinite element  $\psi$  will reduce ("collapse") to the definite element  $v_i$  in a nondegenerate measurement using the  $\{v_i\}$  basis is:

$$\Pr(v_i|\psi) = \frac{\langle \psi|v_i \rangle \langle v_i|\psi \rangle}{\langle \psi|\psi \rangle}.$$