What Counts as a Newtonian System?: The View from Norton's Dome

Samuel C. Fletcher*

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Abstract

If the force on a particle fails to satisfy a Lipschitz condition at a point, it relaxes one of the conditions necessary for a locally unique solution to the particle's equation of motion. I examine the most discussed example of this failure of determinism—that of Norton's dome—as well as two of my own. The best objections to the former hinge on the fact that the dome uses idealizations like perfect constraints to underwrite the failure of determinism. The best objections to my examples, however, are generally very different, with the exception of the Lipschitz condition, which I diagnose as the source of the failure of determinism. Unlike Norton, however, I do not seek to conclude that these examples are necessarily strong evidence that classical mechanics is not deterministic; rather, I want to emphasize the legitimacy of pragmatic considerations in deciding what legitimately counts as a Newtonian system.

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One owes Mr. Lagrange for having shown, in a clear and precise manner, what the special solutions in the theory of curves are; but no one I know has yet proposed to determine their employment in dynamical questions. This is, however, a point of the science which deserves to be completely clarified, and on which we are far from having lifted any kind of difficulty.[1]

(M. Poisson (1806) [18] pp. 62–63)

1 Introduction

Classical mechanics is often held as the paragon of a deterministic theory, but there are in fact many ways in which determinism can fail in it. Some involve supertasks of denumerable particles interacting; others, like the "space invaders" scenario, involve massive particles traveling under their mutual gravitational influence appearing from spacial infinity at an arbitrary time [4, 5]. But perhaps the simplest class of examples consists in (finite) classical systems whose equations of motion are not uniquely fixed by the specification of their initial conditions. Such examples have been rediscovered several times in the literature, and the most recent and concrete example of this is Norton's dome [16]. In the example of the dome, a point mass at rest at the apex slides down acausally at some arbitrary time. Norton's example has generated considerable controversy; but instead of arguing to reject it as illegitimate like many commentators, Malament has brought attention to an underlying vagueness in the definition of a Newtonian system [15].[2] For him, the answer to the question of determinism in classical physics depends on what exactly one takes a Newtonian system to be.

I agree with this conclusion, and this paper is in the same spirit, being an attempt to carry out its implications to the fullest. First, I show that the disagreement between Norton and his interlocutors ultimately rests on differing accounts about what counts as a proper Newtonian system. For each conception thereof, one can answer the question of determinism unambiguously because the mathematical structure of the theory is thereby clear. But, I will stress, there is no *transcendental* reason to choose a sole one among these. In practice, the choice of a particular formulation of classical mechanics will depend largely on pragmatic factors on what one is trying to do with the theory. Instead of conceiving classical mechanics as some canonical theory with a fully delineated subject matter, I picture it as a set of tools

[1] "On doit á M. Lagrange d'avoir montr e, d'une manière claire et précise, ce que représentent les solutions particulières dans la théorie des courbes; mais personne, que je sache, ne s'était encore proposé d'en déterminer l'usage dans les questions de dynamique. C'est cependant un point de la science qui mériterait d'être complètement éclairci et sur lequel nous sommes loin d'avoir levé toute espèce de difficulté." All translations are original, unless otherwise indicated.

[2] I will use "classical" and "Newtonian" as synonyms unless otherwise specified.

from which one selects the hammer or wrench one judges to be most effective for one's purposes, whatever they may be.

My plan is as follows. I begin in §2.1 by reviewing how determinism can fail if the forces acting on a dynamical system do not satisfy a certain continuity condition called *Lipschitz continuity*. Then in §2.2 I recapitulate the details of Norton's dome, explicating some of its (perhaps unapparent) mathematical features. I will take a brief detour in §3 to narrate some of the history of these kinds of indeterministic systems since they were first discovered, before discussing Norton's replies to the example's most pertinent objections in §4. This will aid us in understanding the role of commentators' different perspectives of classical mechanics as we consider their objections. §5.1–5.2 contain two of my own examples as well as a case for why they avoid some of the difficulties of Norton's dome. In particular, I find that the best objections to Norton's and my examples differ in character significantly enough that there is little to unite except for some bare—but important—mathematical features. As I state in the concluding section (§6), however, I do not wish to draw from this that Newtonian mechanics fails determinism once and for all; rather, I wish to emphasize the pragmatic considerations that might force us to take one side or another in the debate.

2 Building Up the Dome

2.1 Setting the Ground

I will say that a given system is *deterministic* or *satisfies determinism* if and only if its state, whose dynamics are described by a set of differential equations, has a unique solution to its initial value problem. For example, in Newtonian mechanics, the equation of motion $\vec{r}(t)$ for some point particle of mass m satisfies

$$\vec{F}(\vec{r}) = m\frac{d^2\vec{r}}{dt^2}, \tag{1}$$

where $\vec{F}(\vec{r}) : D \to \mathbb{R}^3$ (with $D \subset \mathbb{R}^3$) is the force on the particle. If in addition the equation of motion satisfies some initial conditions, e.g.

$$\vec{r}(t_0) = \vec{r}_0, \tag{2}$$

$$\frac{d\vec{r}}{dt}(t_0) = \vec{v}_0, \tag{3}$$

then a theorem from the theory of ordinary differential equations guarantees that there is a unique maximally extended solution to the equation of motion, provided that the force $\vec{F}$ is

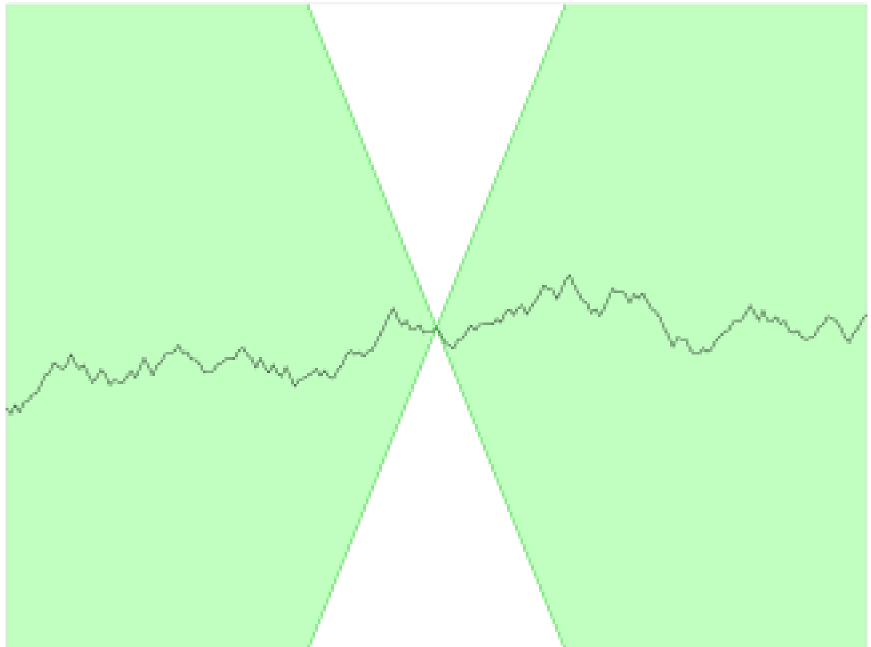


Figure 1: If a function is to be Lipschitz at a point, the graph of the function must lie within the green cones whose slopes are determined by the Lipschitz constant K.

locally Lipschitz continuous on its domain D. Explicitly, we say F is *Lipschitz continuous*, satisfies a *Lipschitz condition*, or simply is *Lipschitz* on its domain D if and only if there is a constant $K > 0$ such that $\forall x, y \in D$, $|F(x) - F(y)| \leq K|x - y|$. So F is *locally Lipschitz continuous* when, for every $x \in D$, there is a neighborhood U of x such that the restriction of F to U is Lipschitz continuous. Intuitively, Lipschitz continuity restricts the slope of the tangent line to a curve from getting too large. (See fig. 1.)

So a system consisting of a massive point particle undergoing Newtonian dynamics (eq. (1) via a locally Lipschitz force is deterministic insofar as the specification of the state of the particle in question—here, its position and velocity—at a particular time t_0 locally fixes the history of the particle, i.e. its state at each time t in a neighborhood of t_0.[3] But what happens when not all of the above conditions are met? Relaxing the continuity condition so that F is non-Lipschitz at a single point can yield uncountably many solutions to eq. (1) satisfying the specified initial conditions. For example, set $\vec{r}_0 = \vec{v}_0 = \mathbf{0}$, and take

$$F(r) = mb^2 r^a \tag{4}$$

to be a radial force with $0 < a < 1$ and b a constant whose units depend on the choice of

[3]The locality of this kind of determinism can be highlighted by reference to “space invader” scenarios, in which the specification of the initial conditions of the system in question fixes its history for only a finite period of time [4, 5]. But this locality as such will not play a role in the examples of this paper.

a; eq. (4) is non-Lipschitz at the origin.[4] Then for the equation of motion there is both a trivial solution,

$$r(t) = 0, \tag{5}$$

and a nontrivial one:

$$r(t) = \left(\frac{(1-a)^2}{2(1+a)}\right)^{\frac{1}{1-a}} (b[t-T])^{\frac{2}{1-a}}. \tag{6}$$

Note that T is any time *whatsoever*. The two solutions given by eqs. (5,6) can be combined in the following way:

$$r(t) = \begin{cases} 0, & \text{if } t \leq T, \\ \left(\frac{(1-a)^2}{2(1+a)}\right)^{\frac{1}{1-a}} (b[t-T])^{\frac{2}{1-a}}, & \text{if } t > T \end{cases} \tag{7}$$

Thus, according to eq. (7), the point particle sits at the origin until the arbitrary time $t = T$, the last time at which it is at rest; afterwards, it moves away as a power of the time elapsed from $t = T$. Because the theory provides no additional information about what the parameter T should be, there is a nondenumerable infinity of possible solutions corresponding to each possible value of $T \in \mathbb{R}$, and determinism fails. I will call this kind of failure of determinism *Lipschitz indeterminism*, and systems that display it *Lipschitz indeterministic*.

2.2 Norton's Dome

The case that Norton considers in [16, 17] is a cylindrically symmetric dome (see Fig. 2) on which a point particle of mass m slides under the influence of a uniform gravitational field. The shape of the dome acts as a kind of constraint surface that yields the necessary force equation for Lipschitz indeterminism. For, suppose the particle is constrained to move on the surface, and the equation for the loss of height h from the top of the dome, as a function of the distance traversed along the surface r, is given by

$$h(r) = (2b^2/3g)r^{3/2}, \tag{8}$$

with g being the acceleration due to gravity and b a constant. Then the force propelling the particle down the slope is the component of the total force tangential to the surface:

$$F_{||} = mg\sin\theta = mg\frac{dh}{dr} = mb^2\sqrt{r}, \tag{9}$$

[4] To see why, suppose otherwise; then there must be some constant $K > 0$ satisfying $|F(r)| \leq K|r|$. But any choice of r satisfying $0 < r < (mb^2/K)^{1-a}$ yields a contradiction.

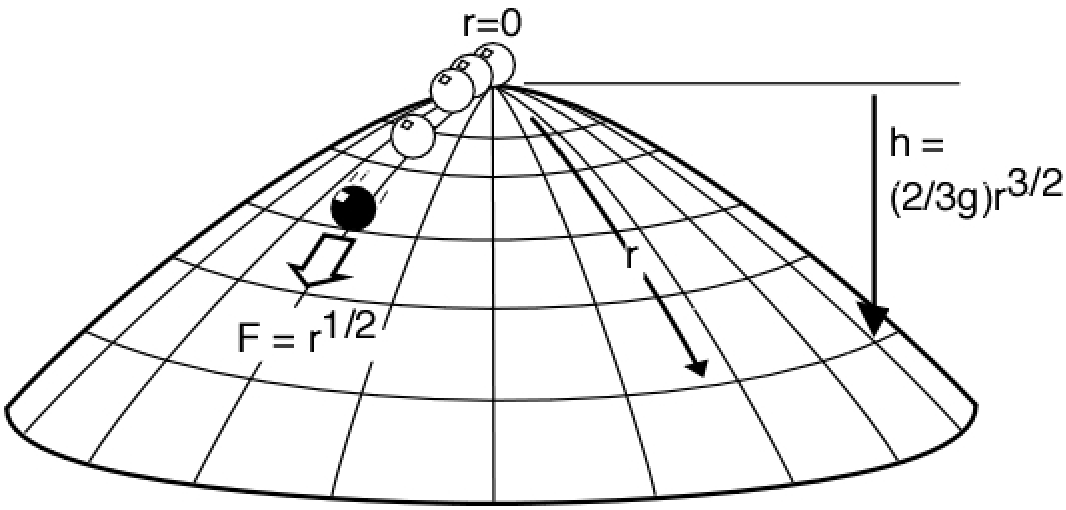


Figure 2: A cartoon of Norton's dome from [16], not to scale and with the constant b set to unity in the appropriate units.

where θ is the angle between the radial tangent to the surface at the particle's location and the the horizontal. Placing the point particle at rest on the apex of the dome yields the Lipschitz indeterminism outlined in §2.1 if one picks $a = 1/2$ in eq. (7) to simplify the equation of motion to

$$r(t) = \begin{cases} 0, & \text{if } t \leq T, \\ \frac{1}{144}(b[t-T])^4, & \text{if } t > T, \end{cases} \tag{10}$$

and the force equation (4) to

$$F(r) = mb^2\sqrt{r}. \tag{11}$$

Now, in the case of the dome, the particle at the apex may fall down after any time, and in any of the radial directions. The reason why the apex of the dome is so special is that it is only at this point that the force acting on the particle is not Lipschitz. One way to see this is that the *curvature* of a cross-section of the dome through the origin diverges as $r \to 0$ [15]. One can understand this by "fitting" a circle along each point of the surface of the dome; as one approaches $r = 0$, the radius of the circle gets smaller and smaller—meaning that a particle traversing its surface undergoes a greater and greater instantaneous angular change—until it becomes zero, and the circle becomes a point. The Lipschitz condition is violated here because the path cannot be contained within any lines with finite slope—a finite rate of instantaneous angular change.

Not only is the initial position special in the respect, but the initial velocity is as well. Malament has shown that for any arbitrarily small velocity in the radial direction, the point particle looses touch with the surface of the dome, and there is no Lipschitz indeterminism [15]. Gravity is the force pushing the particle into the dome—which pushes back—

constraining it to that surface. But for any non-zero initial velocity, the parabolic arc traced by the particle in free fall extends past the dome; the dome falls away from particle faster than the particle falls under gravity, for *any* such initial velocity.

Finally, Norton emphasizes that classical mechanics gives no guidance as to any possible probability distribution over the possible allowed equations of motion.[5] Even relying on symmetry to assign a uniform probability will not work since there is such distribution for the parameter T, which can take on any real value. One might be tempted to assign a probability distribution, in analogy with radioactive decay, where the rate of "decay" is uniform over time. But since such processes require a time constant to define exactly what this rate is, there still remains a parameter totally unspecified by the theory.

3 A Brief History of Multiple Rediscovery

I have kept this section to a minimum for now. One can see [3] for a review.

Poisson published the first investigation into dynamical equations that are not (what we would now call) everywhere Lipschitz continuous [18]. He looked at equations of the form $dv/dt = -av^n$ and $d^2x/dt^2 = ax^n$, the latter we can recognize as a generalization of the dome case. Both can fail Lipschitz continuity. (As to their evolution, he came to much the same conclusions as Zinkernagel [24].)

In the mid-nineteenth century, Duhamel and Cournot published textbooks on the subject, followed by Boussinesq in the 1870s, which was republished through the early 1920s in the dawn of the quantum theory. Boussinesq in particular, however, came to see these systems at the origin of human free will.

In the early 1990s, a series of articles by Hutchison on time-reversal invariance in classical mechanics revived cases of the functions studies by Poisson [7, 8], but it seems few paid attention to them as many have with Norton's dome. The likely reason is that he did not give a physical model for the hypothetical forces.

[5]Although it makes little difference for what follows, I think it would be a mistake, as it appears Norton has done here, even to consider interpreting Lipschitz indeterminism as a species of probabilistic evolution, much as one gets in classical stochastic systems or objective collapse interpretations of nonrelativistic quantum mechanics. The point is simple: classical mechanics simpliciter does not have the mathematical structure of such a theory. Of course, one can filigree classical mechanics with stochastic elements, which may be a legitimate and pragmatic response in appropriate situations. (See the discussion in §6.)

4 Attempts to Demolish the Dome

Norton [17] considers a variety of objections to his example, classifying them as such:[6]

1. The dome incorrectly or incompletely applies Newtonian mechanics.
2. The system consisting of the dome and point particle is in some crucial sense *unphysical*:
   (a) as overdescription, i.e. as a gauge quantity;
   (b) as contradictory, i.e. as contradicting other generally well-established claims of the theory;
   (c) as false, i.e. as empirically falsified;
   (d) as underdescription, in the sense of it indicating the theory's incompleteness.
3. The analysis of the dome uses improper idealizations.

Many of Norton's responses to these categories of objection are cogent, and I will address them in turn.

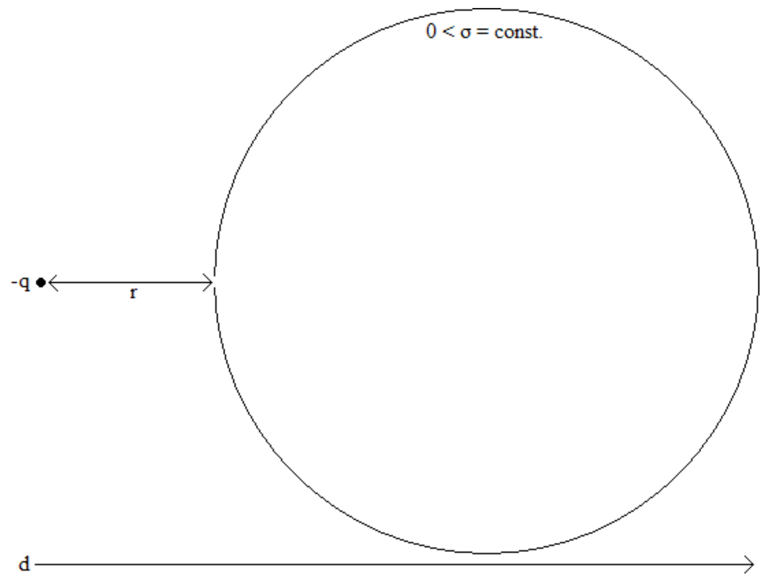
4.1 Objection: Incorrect or Incomplete Application of Newtonian Theory

For the first category—that the principles of Newtonian mechanics have been incorrectly or incompletely applied—he admits that there may be an added constraint or a modification to an accepted rule of the theory, which will disqualify his dome as a legitimate system. For example, consider an objection like this: We can avoid the failure of determinism inherent in eq. (6) and preserve the obviously deterministic solution of eq. (5) by enforcing that all equations of motion be smooth.[7]

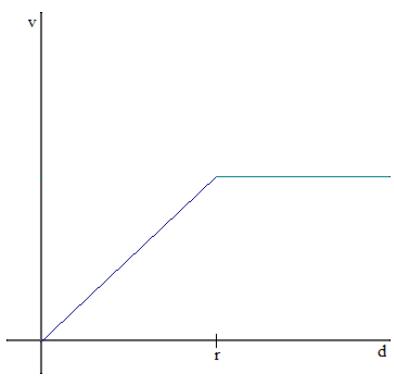
But such a rule unduly restricts the set of systems amenable to analysis. One simple example of this is that of a uniformly charged spherical nonconducting shell of total charge Q and radius R with a pinhole (i.e., a point) removed, and a test particle of opposite charge $-q$ placed outside of the shell on the ray extending from the center of the shell through the pinhole. (See fig. 3.) Released from rest, the test charge will accelerate toward the center of the shell until it is inside, at which point it will stop accelerating because the electric field from the shell vanishes inside. The acceleration is discontinuous so the equation of motion cannot be smooth.

[6] We need not view this classification as a partition of possible objections, since some *particular* objection may inseparably include aspects from more than one category. Nevertheless, viewed as predicates, we can still use them pragmatically to organize the character of possible objections.

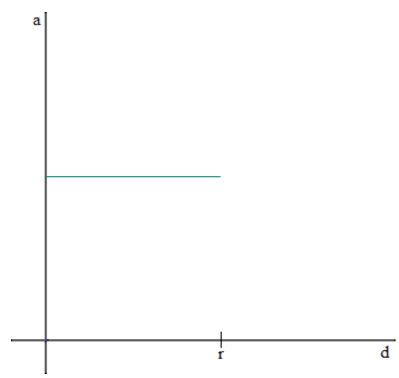
[7] A function f is smooth when its derivatives $\frac{d^n f}{dt^n}$ exist and are continuous for each $n \in \mathbb{N}$.



(a) Shell and point particle.



(b) Velocity vs. position.



(c) Acceleration vs. position.

Figure 3: A cross-section of the uniformly charged spherical nonconducting shell, with graphs of the velocity and acceleration of the point particle as a function of distance from its initial position.

Koralev initially took this as one approach to throwing out the dome, arguing not for smoothness, but for (the weaker) Lipschitz continuity as an implicit assumption of classical mechanics [11]. This seems more or an ad hoc addition than smoothness, since one considers it only to rule out a Lipschitz indeterminism explicitly [5]. Moreover, as Norton points out, Lipschitz continuity is a sufficient but not necessary condition for the uniqueness of solutions to the equations of motion. Thus, requiring it can rule out systems that are not Lipschitz indeterministic, just as enforcing smoothness ruled out textbook examples above. Indeed, any such addition will have to be answerable to systems with an already accepted analysis; i.e., the addition should not be so strong as to disallow or alter the dynamics of systems whose the analysis was previously unproblematic.

Ironically, in the manuscript version of his paper, Korolev considers some such situations that arise in fluid dynamics, whose proper treatment is an active field of research [11]. Without going into detail about such physical systems, I can show how there will always be difficulties and a certain degree of arbitrary stipulation if one seeks to rule out Lipschitz indeterminism through enforcing some stronger form of continuity. There are many options one could take in ruling out the dome. Here are a few, in roughly decreasing strength:

1. Eliminate any system where at least one trajectory in phase space is Lipschitz indeterministic.

2. Given some system, remove any trajectories in phase space that are Lipschitz indeterministic.

3. Given some trajectory of a system in phase space, remove any points that result in Lipschitz indeterminism.

Each of these options perhaps shares some proportion of being too strong and of being ad hoc. For example, the first option rules out the dome entirely, even when one does not consider a particle initially at rest at the apex. The second disallows one from considering (only) incomplete trajectories along the surface of the dome, while allowing trajectories that come arbitrarily close. The third option is the most austere, removing only a single point of phase space from the dome example. It may still seem as hoc, however, if one considers that this move allows for a particle to be located at the apex of the dome for any non-zero velocity. Moreover, none of these options can accommodate forcing the particle at rest at the apex to remain there at rest. (Cf. §3 above.) Nevertheless, they remain options of mathematically kosher additional criteria that can impose.

Zinkernagel, on the other hand, has recently argued not that Norton has neglected a tacit assumption of Newtonian mechanics, but that he has incorrectly applied Newton's first law, that "Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it" [24]. When properly understood, he claims, the correct path of the particle at rest at the apex of the dome is to remain there at rest. Moreover, attempts to reformulate the first law as a logical consequence of the second—the familiar $F = ma$—impoverish the theory of its physical justifications for inertial frames and definition of time.

His argument hinges on an analysis of "continues" and "uniform" in the first law. He reads "continues" as meaning that every change must have a first cause, so that if a particle is at a uniform velocity at time $t = T$, then it must remain at that velocity continuously until at least $t = T + \epsilon$, for some $\epsilon > 0$. This criterion selects the particle-at-rest trajectory for Norton's dome, since it effectively requires the net force acting on any particle to be non-zero only on closed intervals or half-open intervals of the form $[a, b)$, for $a, b \in \mathbb{R}$ and $a < b$. (His reading of "uniform" motion (at some time $t = T$), which he takes to mean that $\lim_{t\uparrow T} \vec{v}(t) = \lim_{t\downarrow T} \vec{v}(t) = \vec{v}(T)$, i.e. the limit from above and below are equal to each other, which imposes no additional constraint.)

At first this seems to be a puzzling response. As Zinkernagel readily admits, this entails that inertial motion cannot occur at isolated points in time—like for a simple harmonic oscillator in motion as its displacement attains zero—or any closed interval, for that matter. But surely there are innumerable innocuous Newtonian systems where forces turn on smoothly from zero to non-zero magnitude, so that there is no first instant when the force causes a change. In a note added in proof, he acknowledges this difficulty, but claims that he can circumvent it by considering any such continuously varying force as the limit of a series of discrete forces with first instances, viz. first causes, which he considers to be more fundamental for Newton. Essentially, this is Euler's method for approximating the solutions to differential equations. He claims that forces that are turned on smoothly can be captured in this framework, while Lipschitz indeterminism cannot.

There is a sense in which he is right: the difference equations (rather than differential equations) of Euler's method are not indeterministic. But this response cannot do. It is well known that Lipschitz discontinuous functions behave erratically under Euler methods, the calculated trajectory of the particle depending significantly on supposedly unphysical choices like the time step size. (For a philosophical discussion of these issues, see [21].)

I will not discuss Zinkernagel's argument that the first law cannot be absorbed into the second because the former is required for the definition of inertia frames and time in Newtonian mechanics. It is clear at this point that Zinkernagel is not arguing about the modern conception of classical mechanics, but about Newtonian mechanics as it was practiced by Newton. He and Norton simply have different subjects of analysis. So, perhaps the former's analysis bears whether Newtonian mechanics *was* deterministic or not. For, as we have seen in §3, Poisson had a similar final analysis of these systems. But that ultimately is not my concern here.

4.2 Objection: the Dome as Unphysical

Norton also has replies to objections whose main thrust is that the dome is somehow unphysical. It is clear that the dome example does not seem to involve overdescription, and the only contradictory claim that it could be seen as making is that it fails determinism, assuming that the latter is an assumption of classical physics. But if the question of determinism is the one under consideration, such as objection is obviously circular.

The two other construals of "unphysical" here provide stronger objections. For example, one might claim that the solutions given by eqs. (5) and (6) constitute underdescription—that only one of the solutions they pick out is correct. If we can rule out all solutions but one, then we would reestablish determinism for the dome. A difficulty arises, however, explaining which solutions are the unphysical ones without depending on an assumed notion of determinism. We might think that Newton's first law—that a body undergoes acceleration only when acted upon by an outside force—suggests that the particle should remain at rest at the apex of the dome. (Cf. Poisson and Zinkernagel above.) Our intuitions about everyday objects (for what they're worth) verify this: we don't expect objects at rest to spontaneously fall into motion. But this intuition is vitiated if we consider the time-reversal the case at hand. We can pick the particle's initial position on the dome and its initial (upward) velocity so that as it traverses the surface of the dome it comes to rest at the apex. The dynamics for the particle here, insofar as they evolve according to the particle's position and velocity, will be identical to that of the original situation in which the particle was placed originally at rest at the apex.

Perhaps one could appeal to empirical evidence as the arbiter—that is, dismiss the solution that fits the data the worst—but because Norton's dome has never been built and tested, there is no such evidence to speak of. In fact, there may never be any evidence that we could produce that bears on Norton's dome. Even if we could construct it perfectly, there is only a single point in phase space—that is, a single pair of positions and velocities—that exhibits Lipschitz indeterminism. Norton responds that this is irrelevant:

> The dome is not intended to represent a real physical system. The dome is purely an idealization within Newtonian theory. On our best understanding of the world, there can be no such system. For an essential part of the setup is to locate the mass exactly at the apex of the dome and exactly at rest. Quantum mechanics

> assures us that cannot be done. What the dome illustrates is indeterminism within Newtonian theory in an idealized system that we do not expect to be realized in the world. [17]

This is a striking declaration. Norton is perspicuous that he is ultimately concerned with the mathematical structure of Newtonian mechanics rather than the theory as it applies to the world.

But not all commentators on the issue feel the same way, especially when they express themselves in physics journals (apparently). For example, Kosyakov considers two kinds of facts to be major barriers to considering cases of Lipschitz indeterminism *compelling* [10]: first, that the forces involved, when posited ab initio, are physically unmotivated (to which we will return shortly), and second, that the initial conditions supporting Lipschitz indeterminism are of measure zero. Because Norton has provided a physical model building from familiar forces and constraints, this first deflationary pin misses. But the second finds its mark so true that it is the primary motivation for much of the rest of Kosyakov's paper, wherein he treats similar examples in only one spatial dimension to ensure that measure is non-zero.

I think it is likely this attitude is common among those for whom Norton describes as accepting his example and analysis of the dome but are unimpressed by it. For these folks, perhaps, the physical theory is interesting and relevant only insofar as it can be used to make testable predictions. For them, the instability of particular features like Lipschitz indeterminism may be sufficient evidence that those features, even if predicted by the theory, are not meaningful. Of course, a detailed account of this will need to argue what this instability amounts to, and such an argument will likely need to tread through idealization.

4.3 Objection: Improper Idealization(s)

Norton admits that there are many idealizations employed by the dome, including: the use of point masses, frictionless sliding, precise initial conditions, infinite curvature at the apex, and complete rigidity in the dome [17]. Korolev has pointed out, in particular, that if the dome were not completely rigid, then no matter how stiff it would still deform sufficiently to prevent Lipschitz indeterminism [11]. He further constructs other examples of Lipschitz indeterminism using only rigid and elastic bodies to avoid the point of infinite curvature of Norton's dome, but finds that they only work when, say ropes or rods are assumed to be either completely elastic or completely rigid. Thus he concludes that infinite idealizations are responsible for the Lipschitz indeterminism [12]. (In my own electrostatic examples considered shortly, this is not a factor, so Korolev's diagnosis cannot be correct.) Norton responds that virtually all of the idealizations he uses are used elsewhere in classical mechanics without complaint or harassment, and Malament concurs [17, 15].[8]

A final option on Norton's part is to posit explicitly a force like eq. (11) so as to avoid the complications of systems of constraints and deforming continua. In particular, the example he considers in [17] involves a fundamental length within which the force acts:

$$F(s) = \begin{cases} m\sqrt{L-r} & \text{if } s \leq L, \\ 0 & \text{if } s > L. \end{cases} \tag{12}$$

This is an attractive force between a fixed (i.e. stationary) "Nortonian" source at the origin and a massive test particle positioned at rest at a distance of exactly L from the source. Shorn of any physically complicating aspects, the only remaining candidate responsible for the failure of determinism is the fact, as before stated, that the force is non-Lipschitz at a point. In my mind there is nothing all that strange about positing force laws to submit for analysis, provided that one does not assuming in general that any such posited function will necessarily satisfy conditions for uniqueness of solution, etc. Conceivably, however, one might object to the outright positing of force laws of an atypical form, especially to derive such unexpected and controversial results.

Laraudogoita is of this opinion:

> By introducing more or less exotic force functions, a trivial advantage can be taken of the fact that not every differential equation possesses a unique, global solution, to obtain simple examples of indeterminist evolution (see, for instance, [7]). This type of result is no more complex than the typical textbook exercise but it is, unfortunately, the least interesting of its kind. Much more relevant are situations in which the only interactions permitted are those specific to Newtonian mechanics, i.e. gravitation and/or collisions (elastic) . . . [14]

The idea seems to be that, if one is going to derive broad metaphysical conclusions from a theory, one must use the "fundamental" forces or interaction of that theory, not merely ones that can be treated theoretically. Callender, in defending the time-reversal invariance of classical mechanics against dissipative forces, used a similar argument: because we have good reason to believe that, at bottom, all forces are conservative, it is very likely that dissipative forces are merely phenomenological—if we could keep track of every molecular collision, then we wouldn't have them [2]. We should only be interested in actual forces that arise from the actual ontology of the theory. And certainly, gravitation, particle scattering, rigid bodies,

[8] It seems the isolated initial condition, of measure zero, where Lipschitz indeterminism occurs is an exception, but I have mentioned this in an earlier section.

and electrostatics are all good candidates for classically fundamental interactions, while the "funny force" of eq. (12) is not. This is why, intuitively Norton's dome is so much more compelling that a model "funny force."

But on the other hand, Wilson has questioned whether or not one can consistently divide classical mechanics into fundamental and phenomenal forces [22]. Instead, he has argued that what is commonly referred to as "classical mechanics" is actually three separate sets of tools that often give mutually incompatible interpretations of the same issue: point particle mechanics, the mechanics of rigid bodies and perfect constraints, and continuum mechanics. Each of them has descriptive gaps that reach all the way to their cores, and appeal to one from another can locally give one a a way to paper over those gaps, but it cannot be done globally. Wilson used the metaphor of a stool with six uneven legs; as it gets pushed around it might settle onto some set of legs or another, but it cannot settle onto all of them.[9]

Nevertheless it is quite easy to construct a situation in which determinism fails using standard, accepted forces without the constraints that yielded so many objections with Norton's dome. The key is realizing that the force equation for a particular body arises from the application of a general force law to a configuration of bodies. If we are prejudiced against introducing queerness into the force law, it is often the case with a sufficiently "robust" force law that we can still have determinism fail by instead shifting that queerness to the configuration of bodies. In §5.1–5.2, I use some examples from electrostatics—i.e. only the Coulomb force[10]—to generate the same kind of failure of determinism.

5 Examples from Electrostatics

5.1 Sphere with Point Charge

Consider a spherically symmetric charge distribution that vanishes beyond some finite radius R (as in fig. 4):

$$\rho(r) = \begin{cases} \frac{5c\epsilon_0}{2\sqrt{r}}, & \text{if } 0 < r \leq R, \\ 0, & \text{if } r = 0 \text{ or } r > R. \end{cases} \tag{13}$$

[9] I am quite sympathetic with Wilson's account, but I do not think he get the analysis of the dome quite right, for he seems to attribute its indeterminism to a mixing of point particle mechanics and the physics of rigid bodies when one employs the reactive/active force decomposition. Roberts has shown, however, that this analysis is neither necessary nor sufficient for indeterminism.

[10] The Coulomb force is the classical force between electrically charged bodies at rest. If r is the distance between bodies of charge q_1 and q_2, then they experience a force with magnitude $F = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}$.

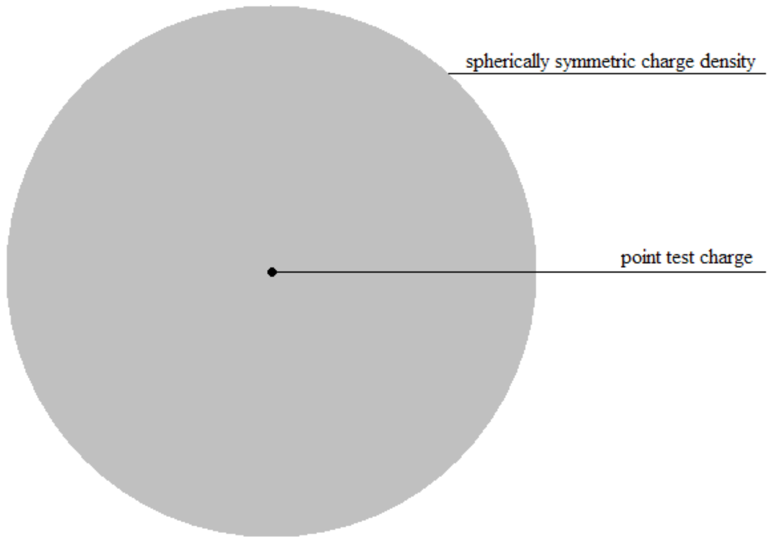


Figure 4: A cross-sectional cartoon of the spherically symmetric charge distribution, with the point test charge at the origin.

Here c is a dimensional constant and ϵ_0 is the permittivity of free space (as usual).[11] Note that while $\lim_{r\to 0^+} \rho(r) \to \infty$, the total charge in the distribution is finite:

$$\begin{aligned} Q &= \int_0^R \int_0^\pi \int_0^{2\pi} \rho(r')(r')^2 \sin\theta' dr'\, d\theta'\, d\phi' \\ &= 10\pi c\epsilon_0 \int_0^R (r')^{3/2} dr' \\ &= 4\pi c\epsilon_0 R^{5/2}. \end{aligned} \tag{14}$$

Utilizing the distribution's spherical symmetry, we can use Gauss' Law to find the radial (and only nonvanishing) component of the electric field for when $r \leq R$:

$$\begin{aligned} E_r(r) &= \frac{1}{4\pi\epsilon_0 r^2} \int_0^r \int_0^\pi \int_0^{2\pi} \rho(r')(r')^2 \sin\theta' dr'\, d\theta'\, d\phi' \\ &= \frac{5c}{2r^2} \int_0^r (r')^{3/2} dr' \\ &= c\sqrt{r} \end{aligned} \tag{15}$$

This is precisely the form of the force law that we need to replicate the behavior from

[11] Norton had found this particular example independently in the latest manuscript version of Norton [17], and includes it in the published version. Previously Zimba [23] developed a somewhat similar electrostatic example using an ideal quadrupole and a spherical charge distribution with a pair of conical sections removed. The latter was done to avoid any objections due to "collisions" between the charge distribution and the point particle—an objection, as I outline below, I think is weak—but as a result the force on the point particle is proportional to $\sqrt{|z|}$ only to first order. Thus Zimba's example is not clearly one of Lipschitz indeterminism.

Norton's dome. For if we now consider a test particle of charge q, it experiences a radial Coulomb force

$$F_r(r) = qE_r(r) = qc\sqrt{r} \tag{16}$$

when $r \leq R$ and, placed at the origin, determinism fails in the same way as before.

Although some of the objections to the sphere example are similar to those against Norton's Dome, many are different, in accordance with the fact that objections tend to be focused on the specific details of the example. The most pertinent ones are outlined below.

Objection 1 (Incomplete physics)**.** *There might be some constraint or alterations to the usual laws of classical mechanics that would prevent this failure of determinism.*

Reply. This is identical to the objection to Norton's dome. In fact, this will be an objection to any example whatsoever, since one can always try to repair determinism by putting new constraints on classical mechanics' domain of applicability. Having previously tried demanding the equations of motion to be smooth, we can instead require that all forces be locally Lipschitz. This *does* effectively reassert determinism in both Norton's dome and the sphere example (and any other example, for that matter) by proscribing the points in phase space necessary for the failure of determinism to occur. However, there are many different ways to implement such a restriction. In roughly ascending strength, we can, for example: restrict the initial conditions of systems to points at which the force is locally Lipschitz; remove from consideration any points in phase space at which the force is not Lipschitz; disallow any trajectories that intersect these points; or limit ourselves to considering only system for which the forces on the dynamical particles are locally Lipschitz on their whole domain. None seems to have any *a priori* considerations above any other, and I would submit that, if one were to take up this objection, such considerations must necessarily arise out of the particularities of the specific system one is modeling. For example, if one is interested just in modeling behavior in a neighborhood of the initial conditions, the weakest condition above might be appropriate, whereas disallowing trajectories that intersect points where the force is not Lipschitz might be best for modeling the extended path of particles. We will return to these considerations in the conclusion.

Objection 2 (Rigidity)**.** *The point particle would immediately collide with the charge distribution as soon as it began to move.*

Reply. This is an objection about *physicality.* But while Newtonian mechanics does purport to describe the physics of rigid bodies, it by no means *limited* to describing them. It is not required that charge distributions also impose constraints on the motions of particles. In other words, we can consider the dynamics of the system as one of point particles rather than of constrained rigid bodies. This is not as unusual as it seems; historically, the Thomson "plum pudding" model of the atom involved a similar kind of charge distribution with a point particle (or several point particles) within. And in galactic astrophysics, it is quite common to consider massive dust that acts like a fluid, with particles passing through it, modeling the distribution of stars or galaxies as such. Closer to home, a classic textbook problem in freshman mechanics is to consider a point particle dropped through a straight "hole" drilled through the center of the Earth—where the scare quotes indicate that the fact that the hole itself, as an absence of mass, is always ignored.[12] Also, the objection only proscribes a failure of determinism when the sphere is perfectly rigid; for if it is allowed to be deformed in the slightest by the test charge, then that test charge would have already instantiated a failure of determinism by moving to cause the deformation. The significance, I think, is that questions about rigidity are not related to determinism failing for a given physical system.

Objection 3 (Continuity)**.** *The charge distribution manifests a pathology, since* $\rho \to \infty$ *as* $r \to 0$*, while* $\rho(0) = 0$*.*

Reply. This is an objection about *idealization*, either about the divergent limit or about the fact that the distribution is discontinuous (or both). However, both features show up commonly in previous accepted and unproblematic example of electrostatics. First of all, there is a sense that point particles, whose charge distribution is given by the Dirac distribution, already manifest both of these supposed pathologies. Beyond that, any considerations of charged (one-dimensional) lines, (two-dimensional) shells or (three-dimensional) perfect conductors will involve discontinuous charge distributions, the former two because of their dimensionality and the latter because net charge on a perfect conductors will distribute itself on the conductor's boundary.

There are also many other physical systems in electrostatics in which the theory states that the charge density diverges. For example, consider a charge Q placed on a perfectly conducting circular disk of radius a centered at the origin. The charge will distribute itself on the disk according to the surface density

$$\sigma(r) = \frac{Q}{2\pi a\sqrt{a^2 - r^2}} \tag{17}$$

for $r < a$ [9].[13] Here the charge density diverges at the edge of the disk.

[12] The particle then undergoes simple harmonic motion. See, for example [19].

[13] An uncannily careful reader might observe that, to be precise, the derivation of this distribution involves taking a disk whose thickness is negligible compared with its radius; then half of eq. (17) gives the charge distribution on each side.

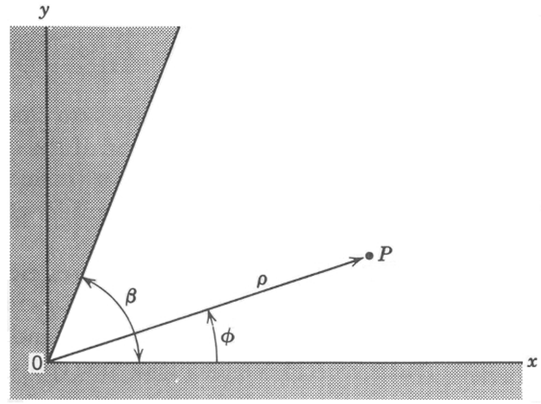


Figure 5: Cross-sectional view of the intersection of two conducting planes at an angle $0 < \beta \leq 2\pi$; if we imagine these planes to be sides of a conducting box, then for $\beta < \pi$ it constitutes an "inside" corner and for $\beta > \pi$, an "outside" corner. [9]

Another example involves the charge density at the corner of a grounded conducting box formed by the intersection of two planes at an angle $0 < \beta \leq 2\pi$, as in Fig. (5). Symmetry provides that the charge distribution on each plane be equal; for each plane, the surface charge density as a function of the distance r from the corner is

$$\sigma(r) = -\sum_{m=1}^{\infty} \frac{m a_m}{4\beta} r^{(m\pi/\beta)-1} \approx -\frac{a_1}{4\beta} r^{(\pi/\beta)-1}, \tag{18}$$

with the last equation the dominant behavior near $r = 0$, and each a_m is a constant determined by the electric potential far away from the corner [9]. Notice that for $\beta > \pi$, which represents the cases for the "outside" corner of a box, the charge density diverges at the corner.

5.2 Sphere with Ideal Dipole

We now consider an example whose setup is analogous to the first except the form of the spherically symmetric charge distribution is different, and the point charge will be replaced by an ideal dipole.[14] So let the charge density be given by

$$\rho(r) = \begin{cases} c\epsilon_0\sqrt{r} & \text{if } r \leq R, \\ 0 & \text{if } r > R. \end{cases} \tag{19}$$

[14] An ideal dipole is a formed as follows. Take two point charges q and $-q$, and let $\vec{d}$ be the displacement vector from $-q$ to q. Define the *dipole moment* as $\vec{p} = q\vec{d}$, then take the limit as $q \to \infty$ and $||\vec{d}|| \to 0$ while holding $\vec{p}$ constant.

The electric field, by symmetry, is nonvanishing only in its radial component. For $r \leq R$, then,

$$\begin{aligned} E_r(r) &= \frac{1}{4\pi\epsilon_0 r^2}\int \rho dV \\ &= \frac{c}{r^2}\int_0^r (r')^{5/2} dr' \\ &= (2c/7)r^{3/2}. \end{aligned} \tag{20}$$

Now we introduce an ideal dipole with moment $\vec{p}$. Because of the spherically symmetric distribution, the force on the dipole at radius r is given by

$$\begin{aligned} \vec{F}(r) &= (\vec{p}\cdot\vec{\nabla})\vec{E} \\ &= \frac{p_r}{r^2}\frac{\partial}{\partial r}(r^2 E_r(r))\hat{r} \\ &= \frac{2cp_r}{7r^2}\frac{\partial}{\partial r}(r^{7/2})\hat{r} \\ &= cp_r\sqrt{r}\hat{r}, \end{aligned} \tag{21}$$

where p_r is the radial component of the dipole moment at radius r. There must be some direction in which $p_r = \sqrt{p_x^2 + p_y^2 + p_z^2}$ is nonvanishing for each r, for otherwise we would not have a dipole in the first place. Placing the dipole at the origin, determinism fails for these directions—any direction that is not orthogonal to $\vec{p}$.

While this example may also be vulnerable to the incompleteness and rigidity objections leveled at the original charged sphere, this new charge distribution does not have any of the continuity features that could have been pathological in the previous one. Instead, if one seeks grounds of idealization on which to object, one must focus instead upon the ideal dipole. On the one hand, the limiting condition involved in its definition might be hard to justify physically, even though physicists and chemists model molecules using these kinds of admitted approximations all the time. On the other hand, strictly speaking, the total charge of the dipole is always zero, so there doesn't seem to be ill-behavior on account of $q \to \infty$. Furthermore, in the the context of the theory, it is, mathematically speaking, a legitimate move, and doesn't seem to be any more of an idealization than a mundane point particle. The fact that the theory is being used to approximate a real situation doesn't somehow invalidate what the theory predicts, the latter of which is what we're interested in here.

	Norton's Dome	Test Charge	Ideal Dipole
Incompleteness	Lipschitz	Lipschitz	Lipschitz
Unphysicality	Nonconstructibility	Rigidity	Rigidity
Idealization	Mixing rigid and point dynamics, Surface tension, Escape velocity,...	Pathological ρ	Ideal Dipole

Table 1: A summary of the objections to each of the three examples that fail to satisfy determinism.

6 Conclusion and Prospects

My intent in showing the two examples with the point charge and the ideal dipole was to highlight how, Lipschitz continuity aside, *different* kinds of objections were required for each. (See table 1.) The unphysicality objections—the nonconstructability of the systems in question, and the rigidity of the spheres—were the weakest, since they seemed ad hoc responses to the failure of determinism rather than the systems themselves. And the fact that the charge distribution for the sphere and test charge system diverges as $r \to 0$ seems to be just an accidental fact about inverse square force laws; the dipole case eliminates this problem since it couples to the electric field with an extra factor of r in the denominator. The fact that an ideal dipole might be controversial seems to be an accidental fact about electrostatics rather than a "cause" of the failure of determinism. Objections like these that fall under the banner of "illicit idealizations" also seem to be missing the point; they are only proffered in cases to prop up determinism, but not in the numerous other cases from physics that do not threaten to fail to be deterministic.

Does this mean that Newtonian mechanics is indeterministic *simpliciter*? No. And similarly we must reject the naive position that it is fully and without qualification deterministic. In my mind it is more useful to view Newtonian mechanics as a kind of metatheory, a framework or a language that the classical physicist uses to develop particular theories—gravitation, say, or electromagnetism—that model particular systems of interest. Consequently we may impose continuity conditions on forces according to our purposes, restricting the domain of application of our model, or modifying its behavior, according to the pragmatic concerns of system at hand. Surely Norton [17] would reply that this seems to be a *post hoc* move, since Lipschitz continuity is virtually never cited as an actual constraint on systems to be analyzed. But I would contend that physicists rarely bother to make explicit these kinds of constraints that, on the face it, seem to apply only to nuisance cases.

For example, in the cases of Newtonian gravitation and classical electrodynamics—both of which have inverse square force laws—the fact that the force is undefined for all particle configurations in which two particles share the same position is rarely mentioned, and computer simulations of these kinds of systems will often employ a workaround, such as giving the particles a small but finite radius and letting collisions be perfectly elastic. To see how these cases often go unnoticed, consider the uniformly charged spherical shell given in fig. 3. To calculate the electric field produced by the shell, we can use Gauss' Law, which relates the electric flux through a closed surface to the charge contained in the surface. But it the surface is coextensive with the shell (modulo pinhole), then the notion of enclosed charge is ambiguous. Nevertheless, one can calculate the field inside and outside the shell and provide the dynamics for the point test charge, while "connecting" its trajectory by hand at the boundary of the shell. Because the graph of the velocity then has a kink, the acceleration cannot be related to the velocity as $\vec{a} = \frac{d\vec{v}}{dt}$ in the typical fashion. Yet this unspoken bridging principle does not constitute a failure of classical physics.

In a similar manner, there are many ways that one can enforce the Lipschitz condition if one desires by restricting a model's domain of application, or by changing the dynamics by hand at the appropriate places. Eliminating elements of phase space and inserting new dynamics by hand at these points do not constitute new techniques for classical mechanics, and are no more *ad hoc* that the same techniques used to model, according to one's purposes, classical systems that do not violate determinism.

I hope I have shown that different commentators have taken different versions of these theories and compared them unknowingly. I also hope I have shown why any particular choice of these is not priviledge over any other, since each is chosen to be useful in its own way. I agree, finally, with Wilson that there is no canonical classical mechanics, but think that one can still answer the question of determinism once one has specified with mathematical precision just which one one is talking about. Ultimately, will complicate Norton's project, for there is no privileged hill on which he can stand and make claims about the whole of classical mechanics.

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