Which quantum field theories exist?

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1 Prospectus

We propose a list of five main meanings of the phrase 'well-behaved theory': i.e. infinities are either absent, or appropriately "controlled". For precisely defined theories, two meanings (Section 3.1); for the others, three meanings (Section 3.2). This list develops Section 2.2.1 of Bouatta and Butterfield (2012).

The most interesting contrast is for QFT: between our fourth and fifth meanings: between a theory being renormalizable, and its having an ultra-violet i.e. high-energy fixed point under the renormalization group flow. We will see three kinds of fixed point: 'asymptotic freedom', 'conformal invariance', and 'asymptotic safety'.

This is interesting because the renormalization group framework has two main aspects: the flow between theories, and the flow's fixed points. But the philosophical literature has emphasized the first aspect, at the expense of the second. (This goes along with emphasizing cut-offs in integrals over energies and momenta, at the expense of cut-off-independent features.)

We hope the existence of such points can resolve recent debate between philosophical enthusiasts of algebraic or constructive quantum field theory, and enthusiasts of heuristic or conventional quantum field theory.

2 Preliminaries

2.1 'Theory' and 'precise definition'

The questions:

(Def): whether 'theory' can be precisely defined, faithfully to its root meaning;

(Use): whether 'theory' is a useful category for the philosophical analysis of science.

are ambiguous according to whether 'theory' means:

(Gen): theory in general, or

(Spec): a specific theory.

So one can say 'No' to (Def) and-or (Use) under meaning (Gen)—or be agnostic about these general philosophical questions—while saying 'Yes' to (Def) and-or (Use) for a specific physical theory.

This will be our position. And we will count a theory as precisely defined if its central notions and claims have a rigorous definition by the standards of today's mathematics community.

Of course, several important (and empirically successful) quantum field theories are at present *not* precisely defined (this is due to difficulties in defining path-integrals.)

Here we say 'No' to (Def), but 'Yes' to (Use), under meaning (Spec). That is: even if a specific physical theory is not (or not yet) precisely defined, it can be illuminating to analyse it—hence our three meanings in Section 3.2.

2.2 Infinities in physics

One must distinguish a theory being precisely defined from both:
(a) the avoidance of approximation methods and
(b) the avoidance of infinities.

As to (b): It is a familiar theme in physics that infinity need not spell disaster. A quantity's taking an infinite value can convey a lot of information (of at least a qualitative kind): both

(i): *about* the given theory: e.g. sources and sinks in a classical dynamical system's phase portrait; e.g. representing phase transitions by thermodynamic quantities being infinite; and.

(ii): *about adjacent/successor theories*, which avoid the infinities: e.g. in geometric optics, the caustics (places where the light-rays are infinitely intense) are decorated with structures that reflects how the underlying wave optics avoids the singularities (Michael Berry).

3 Five meanings of 'well-behaved theory'

3.1 Two meanings for a precisely defined theory

[1]: (GlobalFinite): In all solutions of the theory, all quantities at all times have a finite value. The idea is that the situation is as good as you could wish for. The theory is precisely defined, with a precise set of solutions/models; and every solution is well-behaved in the announced sense, that all quantities have a finite value at all times. Example: the harmonic oscillator.

[2]: (LocalFinite): In some solutions of the theory, some quantities at some times have an infinite value.
The idea is that some solution 'looks good locally' but 'blows up in finite time'. Example: general relativity.

3.2 Three meanings for a theory that is not precisely defined

The main trouble is that the central theoretical notion, the path-integral (functional-integral)

$$\int \mathcal{D}\phi \ \exp(iS[\phi]/\hbar) \ , \tag{1}$$

can only be precisely defined for non-interacting field theories. Among the strategies for coping with this, we focus on perturbation series. For this, the root meaning of 'well-behaved' is convergence, cf. [3].

In quantum field theories, convergence usually fails: and even the terms of the series are infinite, thanks to integrating over arbitrarily high energies/momenta. In this sad situation, there are two independent saving graces, each seen in some theories: [4] and [5].

[3]: (Convergence): The power series in the coupling constant g,

$$\sum_{n=0}^{\infty} g^n A_n \ , \tag{2}$$

is convergent: or if it is (as usual!) divergent, it is (sometimes!) summable in some weaker sense, e.g. Borel summability.

But in QFT, we are faced with the A_n being infinite. An expedient is to introduce a cutoff Λ, i.e. a finite upper limit of the energy/momentum integral, yielding $A_0(\Lambda) + gA_1(\Lambda) + g^2A_2(\Lambda) + ... + g^nA_n(\Lambda) + ...$ One way to justify this is:

[4]: (Renormalizability): The parameters (g, the mass, etc.) depend on the cutoff Λ in such a way that each term $g^n(\Lambda)A_n(\Lambda)$ is in fact independent of the cutoff, and finite. (Dyson provided criteria for this to be possible, related to the dimension of g etc.) Example: QED.

Note: a theory could be renormalizable, yet fail to 'exist', in the sense that its correlations are *not* the limits of the values given by a theory on a lattice in space or spacetime, as the lattice-spacing tends to zero.

[5]: (UV fixed point): The coupling constant g is a function of the energy-scale μ at which we describe or probe the system: $g \equiv g(\mu)$. But $g(\mu)$ tends to a finite limit as μ tends to infinity. So there is no dependence on the cutoff. We define the beta-function by

$$\beta(g) := \frac{dg}{d\ln\mu} \equiv \mu\frac{dg}{d\mu}. \tag{3}$$

There are three important cases:

(a): *asymptotic freedom* is: $\lim_{\mu\to\infty}\beta = 0$; $\lim_{\mu\to\infty} g = 0$. Example: QCD;

(b): *asymptotic safety* is: $\lim_{\mu\to\infty}\beta = 0$; $\lim_{\mu\to\infty} g = g_\star \neq 0$. Example?: quantum GR;

(c): *conformal invariance* is: $\beta \equiv 0$ i.e. g is constant, independent of μ. Example: Yang-Mills theory with four copies of supersymmetry.

Broadly speaking, [4] and [5] are logical independent. The consensus in physics is that only theories with [5] will be precisely definable.

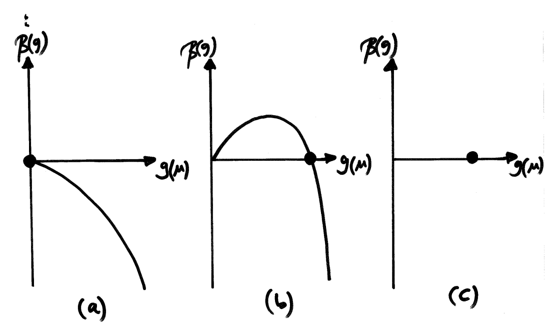

Figure 1: UV fixed points

4 Combatting the pessimistic impression

We said the philosophical literature has emphasized the flow between theories, at the expense of its fixed points. More pointedly: ultra-violet fixed points rebut a misleading impression that:

(i): quantum field theories are endemically badly behaved at high energies; and (ii): the best you could get is a sequence of effective theories, each of which is accurate only up to some energy-scale, and badly behaved above it.

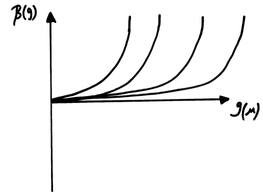

Figure 2: Sequence of effective theories

Agreed: QED probably is badly behaved: i.e. $\lim_{\mu\to\infty} \beta = \infty$. So one should not let the cutoff go to infinity.

But this does not make (ii)'s sequence of *effective theories* compulsory. For QED might have a *UV completion*. And furthermore: QED is not 'the only game in town'. Indeed: QCD, and pure electro-weak theory (without the Higgs) are asymptotically free (as well as renormalizable). And asymptotically free theories that are empirically successful and much studied, such as QCD, probably *can be* precisely defined: though it might take ten years' work by a devoted monk to show it ...

Example showing how too many fermions can spoil the party: For QCD, the one-loop β-function:

$$\beta_1(g) := \mu \frac{dg}{d\mu} = -\frac{g^3}{16\pi^2}\Big(11 - \frac{2}{3}N_F\Big). \quad (4)$$

The minus sign in Eq. (4) means there is asymptotic freedom i.e. g decreases at higher energies, with $\lim_{\mu\to\infty} g = 0$, as long as the number N_F of fermion flavours is small enough (less than 16).

5 Making peace between rigour and heuristics

Our discussion prompts a warning about efforts to mathematically understand the usual Higgs mechanism for spontaneous symmetry breaking within rigorous quantum field theory: an effort enjoined by some philosophical literature. Namely: although the pure electroweak theory (gauge group $SU(2) \times U(1)$) is asymptotically free, the theory with the Higgs boson added in is probably *not*: and so may well *not be* precisely definable.

More generally, it suggests a compromise in the debate among philosophers about whether to 'dive in' to assessing heuristic quantum field theory. Namely: one should not have a uniform stance for all such theories. Agreed, for a theor that probably cannot be precisely defined, e.g. QED: we should treat cautiously its apparent ontological claims, or 'world-picture', for example scrutinizing in what sense it is a field theory. But at least for theories that are well-behaved in sense [5], philosophers should dive in.

Reference: Bouatta, N and Butterfield, J (2012): On emergence in gauge theories at the 't Hooft limit. http://philsci-archive.pitt.edu/9288/