HOW TO TIME REVERSE A QUANTUM SYSTEM

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ABSTRACT. The received view of the meaning of time reversal in quantum mechanics suffers from a problem of conventionality. I review existing attempts by philosophers to avoid this problem, and argue that they fall short. In their stead, I propose an alternative approach to the meaning of time reversal in quantum theory inspired by Wigner. In particular, I show that a refinement of Wigner's assumptions gives rise to several precise theorems beyond what Wigner himself realized, which completely characterize the meaning of time reversal, while avoiding the shortcomings of the received view.

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1. Introduction

The 'received view' of the meaning of time reversal in quantum theory suffers from a problem of conventionality. This problem arises out of two shortcomings: (1) the received view inherits the difficulties of the so-called 'quantization picture' of quantum theory; and (2) the received view does not address the philosophical project of determining *why* the time reversal transformation is defined the way it is. Both shortcomings leave the meaning of time reversal unjustified, and thus apparently 'conventionally' defined. This paper begins with an exposition of these two problems. I then review two attempts by philosophers to resolve them, namely those of Callender (2000) and Albert (2000), and argue that they are unsatisfactory. In their stead, I propose an alternative approach, inspired by the characterization of time reversal of Wigner (1931). In particular, I will show that a refinement of Wigner's assumptions gives rise to several precise theorems beyond what Wigner himself realized, which completely characterize the meaning of time reversal, while avoiding the shortcomings of the received view.

2. The received view of time reversal

When Wigner (1931) introduced the modern approach to time reversal in quantum mechanics, he assumed that all transition probabilities between states in quantum mechanics are time reversal invariant. This is typically expressed by the requirement that if $T : \mathcal{H} \to \mathcal{H}$ is a bijection on Hilbert space implementing time reversal, then for all $\psi, \phi \in \mathcal{H}$,

$$|\langle T\psi, T\phi\rangle| = |\langle \psi, \phi\rangle| \,. \tag{1}$$

This assumption led Wigner to the following important theorem, which has since become central to our understanding of symmetries in quantum theory. In effect, Wigner found[1] that any operator satisfying Equation (1) is either unitary or antiunitary:

Wigner's Theorem. *A bijection on $\mathcal{H}$ that preserves transition probabilities is implemented by either a unitary operator or an antiunitary operator.*

A *unitary operator* U is a linear operator whose inverse is equal to its adjoint: $U^{-1} = U^{\dagger}$. An *antiunitary operator* A is one that is the composition of a unitary operator with a conjugation operator, $A = UK$, where K is any operator satisfying $K^2 = I$ and

[1] For the sake of historical accuracy: Wigner himself expressed time reversal as a bijection $\mathbf{T}$ on the *unit rays* in $\mathcal{H}$. He then showed there exists an induced bijection on Hilbert space $T : \mathcal{H} \to \mathcal{H}$ implementing $\mathbf{T}$ such that, if $|\langle T\psi, T\phi\rangle| = |\langle \psi, \phi\rangle|$, then T is either unitary-linear or antiunitary-antilinear. See (Wigner 1931, Appendix to §20) for a sketch, and (Bargmann 1964) for the classic proof.

$\langle K\psi, K\phi\rangle = \langle\psi, \phi\rangle^*$, for all ψ, ϕ in the Hilbert space[2]. For the purposes of understanding time reversal, Wigner's Theorem is significant because of the following immediate consequence.

Corollary. *If the time reversal operator in quantum mechanics (1) preserves transition probabilities, and (2) cannot be implemented by a unitary operator, then it is implemented by an antiunitary operator.*

This corollary is the starting point for what I call the *received view of time reversal*, which consists in a particular argument for the claim that T is in fact antiunitary. The argument runs as follows.

First, take (1) for granted: the received view assumes that time reversal preserves transition probabilities. Second, assume that quantum observables transform under time reversal like their classical analogues[3]. Robert Sachs expresses this second step in the following (note that in the Sachs notation, a 'prime' denotes application of the time reversal operator: $X' := TXT^{-1}$ and $P' := TPT^{-1}$):

> At the same time, motion reversal imposes the requirements, in accordance with the classical conditions,
>
> $$X' = X, \qquad P' = -P, \qquad \sigma' = -\sigma,$$
>
> since momentum and angular momentum change sign on reversal. (Sachs 1987, p.34.)

The received view then proceeds to observe that, if we apply these classical transformations to the commutation relation $i = [X, P]$, then we can show T cannot be unitary:

$$\begin{aligned}
TiT^{-1} &= T[X, P]T^{-1} \\
&= (TXT^{-1})(TPT^{-1}) - (TPT^{-1})(TXT^{-1}) \\
&= X'P' - P'X' \\
&= -(XP - PX) \\
&= -[X, P] = -i,
\end{aligned}$$

[2] As a simple heuristic, the uninitiated can minimally take from this that antiunitary operators involve complex conjugation, while unitary operators do not.

[3] Recall that classically, if we have a ball rolling to the right, the *time reverse* of this motion is standardly taken to be a ball rolling to the left. The order of events is reversed, all the same positions are occupied, and the velocities of the ball at each moment are reversed. In summary: $t \mapsto -t$, $x \mapsto x$, and $p \mapsto -p$.

where the classical transformation rules indicated by Sachs have been applied in the fourth equality. But if T were unitary, then $TiT^{-1} = iTT^{-1} = i$, which would be a contradiction. So, the time reversal operator T is not unitary. Therefore, by the corollary to Wigner's theorem, T must be antiunitary.

The received view of time reversal is very common among the textbooks[4]. It is also well-known to philosophers of physics. For example, Craig Callender has argued that the antiunitarity of time reversal "is necessitated by the need for quantum mechanics to correspond to classical mechanics" in a limiting case (Callender 2000, p.263). And John Earman has noted that time reversal applied to the position and momentum observables "should reproduce classical results" (Earman 2002, p.248).

Part of the conclusion of this paper is that time reversal is indeed antiunitary, and does indeed reproduce classical transformation rules. However, as we will now see, the *argument* proposed by the received view is inadequate, leaving a significant part of the meaning of time reversal unexplained.

3. Inadequacy of the received view

We have just seen that the essential machinery of the received view of time reversal in quantum mechanics rests on two assumptions. The first is that transition probabilities are time reversal invariant. The second is that time reversal in quantum mechanics must conform to classical transformation rules. The first assumption is already worrisome, according to a well-known perspective expressed by Artzenius and Greaves (which they attribute to David Albert):

> one should *first* work out which transformation on the set of instantaneous states implements the idea of 'the same thing happening backwards in time'; then and only then one should compare ones time reversal operation to the equations of motion, and find out whether or not the theory is time reversal invariant. (Arntzenius and Greaves 2009, p.563.)

One might thus demand that we *derive* the time reversal invariance of transition probabilities, instead of just assuming it. This worry is compounded in the particular case of quantum theory by the fact that, even according to the received view of time reversal, quantum theory is *not* time reversal invariant. For example, it is well-known that flavor-changing weak interactions give rise to processes that are CP-violating, and hence T-violating by the CPT theorem[5]. So, it is not *prima facie* clear how to justify the assumption that

[4]For example, versions of it can be found in (Ballentine 1998, p.378), (Le Bellac 2006, p.556), (Merzbacher 1998, p.441), (Messiah 1999, p.667), (Shankar 1980, p.301-302), and (Tannor 2007, p.124), among others.

[5]See (Sachs 1987, Chapter 9) for an overview.

transition probabilities are time reversal invariant. Absent such a justification, this characterization of time reversal appears to be a purely conventional choice.

The second assumption, that time reversal in quantum mechanics conforms to classical transformation rules, is perhaps even more problematic. First, it inherits the difficulties of the so-called 'quantization picture' of quantum theory; and second, it is incomplete as a characterization of time reversal. Both problems leave the meaning of time reversal ultimately unexplained, and thus apparently conventionally defined. Let us treat each of them in turn.

3.1. **Difficulties with the quantization picture.** The claim that quantum observables time-reverse like their classical analogues belongs to a class of analogies, collectively known as the *quantization picture*. On this picture, one arrives at a correct quantum description by beginning with a classical Hamiltonian system, and applying a series of transformations known as 'quantization' in order to generate a quantum system. One well-known such transformation is the homomorphism $Q : f \mapsto Q(f)$, from the Poisson algebra of smooth functions f of classical variables to the C^* algebra of quantum observables $Q(f)$, given by

$$Q(\{f, g\}) = \frac{1}{i\hbar}[Q(f), Q(g)],$$

where $\{\,,\,\}$ is the Poisson bracket and $[\,,\,]$ is the commutator bracket.

The quantization picture is more than the view that quantum theory must approximate classical theory in appropriate limiting cases; it is the view that "the primary role of the classical theory is not in approximating the quantum theory, but in providing a framework for its interpretation" (Woodhouse 1991, vi). Unfortunately, providing it with rigorous expression has resisted many of our best mathematical attempts. Although the quantization mapping Q exists as a homomorphism for on functions f and g that are quadratic (such as for classical free particles or harmonic oscillators), this mapping does not in general exist. As it turns out, there can be no 'quantization' correspondence between the Poisson algebra of smooth functions of classical variables and the operators in an irreducible representation of a quantum algebra[6]. Of course, an alternative expression of quantization might still exist; but this remains an open problem in the literature.

It would be a shame to rest the foundations of quantum theory on a procedure as tenuous as quantization. Worse, from an interpretive perspective, the quantization picture is entirely backwards. Quantum theory (or something like it) is thought to provide the correct description of the fundamental constituents of the world. In contrast, classical physics is only correct in certain limiting cases, in which it can be shown to derive from

[6]See (Woodhouse 1991, Ch.8-9), (Streater 2007, §12.7) for an overview.

quantum theory. Thus, it seems that our framework for interpreting classical physics should depend on quantum physics, and not the other way around.

The received view of time reversal, in seeking to inject classical transformation rules into the foundations of quantum theory, inherits the very same problems. What is needed, I claim, is an account of time reversal that avoids the appeals to classical physics inherent in the quantization picture. Of course, one can still accept that classical theory is an approximation of of quantum theory. But we would like to do so without introducing a classical framework into our interpretation of symmetry operators.

3.2. **Incompleteness of the definition.** The second problem is that, no matter how we view the formulation of quantum theory, the received view does not provide a complete characterization of how time reversal transforms observables.

A recent debate about time reversal in electromagnetism illustrates what a satisfactory 'complete' characterization of time reversal consists in. One might simply declare that time reversal in electromagnetism reverses the sign of the magnetic field $\mathbf{B}$, while leaving the electric field $\mathbf{E}$ unchanged. But if a declaration is *all* that is required, then the door is left open for non-standard transformation rules such as those proposed by Albert (2000, §1) as well. Many such peculiarities can be avoided through a more complete characterization of time reversal, in which one provides some independent justification for the transformation rules. This is provided, for example, when Malament (2004) and Arntzenius and Greaves (2009) characterize time reversal as a mapping induced when the temporal orientation of spacetime is reversed[7]. One can show that, once some well-motivated assumptions about the time reversal mapping are specified, the transformation rules for $\mathbf{B}$ and $\mathbf{E}$ can be *derived*, rather than merely declared.

Similarly, a complete characterization of time reversal in quantum mechanics should do more than just declare the transformation rules for observables. These transformation rules should be derived from well-motivated principles. The principle that 'quantum observables transform like their classical analogues' will not suffice, since this just shifts the problem to one of determining why the *classical* transformation rules are the way they are. Moreover, even if we ignore this problem, there remain many observables in quantum theory that have no classical analogue. The principle that 'quantum observables transform like their classical analogues' obviously does not apply to such observables. For example, although the orbital angular momentum operator has a classical analogue,

[7] Arntzenius and Greaves (2009, p.564) call the particular character of this justification 'geometric,' in that it involves first designating how each of the observable quantities in the theory is functionally dependent on the temporal orientation t^a of a spacetime $(\mathcal{M}, g_{ab}, t^a)$. I will not require every complete characterization of time reversal be geometric. Rather, a complete characterization should propose *some* well-motivated justification of the transformation rules.

intrinsic spin does not[8]. Neither does the strangeness observable, nor does isospin, nor do a host of other 'non-classical' observables. The received view is silent as to how to time reverse such observables, short of simply declaring them. And if we allow the latter, then the meaning of time reversal is conventional.

In sum, the inadequacies of the received view of time reversal suggests that we have two open problems in the foundations of quantum theory.

Problem 1. Justify the antiunitarity of T in quantum theory without appeal to classical mechanics.

Problem 2. Derive the rules according to which T transforms observables, including those with no classical analogue, from independent principles.

In the next section, I will review the recent attempts by philosophers to overcome one or the other of these problems, and show how they fall short. I will then return to the original approach to time reversal introduced by Wigner, and argue that it leads to a more satisfying resolution of these problems.

4. Three alternatives that fall short

4.1. **Callender's Correspondence Rule.** Craig Callender (2000) has provided one sophisticated response to the problems with the received view of time reversal. Callender argues that the correspondence between the quantum and classical transformation rules for position and momentum in fact derives from Ehrenfest's theorem, and thus from essentially quantum mechanical principles. If this is right, then Callender has made significant progress toward solving both problems above. We can derive the quantum transformation rules, instead of just assuming them, and then proceed to derive antiunitarity from the corollary to Wigner's theorem – assuming, as above, that transition probabilities are time reversal invariant. Unfortunately, this technique still does not provide a justification of the transformation rules for non-classical observables. Moreover, it turns out that the derivation from Ehrenfest's theorem is *not* essentially quantum mechanical; it also requires a classical correspondence rule, although it is not as obvious as that of the received view.

Given some initial state, let $\langle X_t\rangle$ and $\langle P_t\rangle$ denote the time evolution of the expectation values of X and P, respectively[9]. Ehrenfest showed that, to a good approximation,

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[8] Of course, both contribute to *total* angular momentum of a system, but that is not enough to establish that the two quantities time reverse in the same way.

[9] More precisely, given an initial state $\psi(0)$, the time evolution of the expectation value of the X observable is given by $\langle\psi(t), X\psi(t)\rangle$ in the Schrödinger representation, and similarly for P. I abbreviate this time evolution as $\langle X_t\rangle$ and $\langle P_t\rangle$, respectively.

$\langle X_t \rangle$ and $\langle P_t \rangle$ behave classically; that is, they provide a solution to the classical Hamiltonian equations. Now, given a conservative Hamiltonian, it is a purely formal property of these equations that, if the position-momentum pair $\mathbf{x}(t), \mathbf{p}(t)$ forms a solution, then so does the classical time reverse: $\mathbf{x}(-t), -\mathbf{p}(-t)$. Therefore, since Ehrenfest's theorem guarantees the pair $\langle X_t \rangle, \langle P_t \rangle$ forms a solution, so too does the pair $\langle X_{-t} \rangle, \langle -P_{-t} \rangle$, where I have used the fact that $-\langle P_{-t} \rangle = \langle -P_{-t} \rangle$. Callender takes this to imply that, since this classical trajectory and its classical time reverse are lawful solutions to the Hamiltonian equations, then in addition "we would expect the quantum versions of these trajectories – at least on average – to be lawful" (Callender 2000, p.266). He concludes that the momentum operator P must reverse sign under time reversal, while X stays fixed like its classical analogue. This would provide a rather interesting argument for the quantum transformation rules, and hence for antiunitarity.

However, it is important to note that Callender's conclusion is only possible given a hidden appeal to classical mechanics. To see why, let's be a little more careful. There are really two time-reversal operators in play in Callender's argument: a quantum one acting on Hilbert space, and a classical one acting on classical phase space. Call the quantum time reversal operator T, and the classical time reversal operator T^C. Callender's argument tacitly assumes a *correspondence rule* between the quantum and classical time reversal operators. Namely, Callender assumes that for any given state,

$$\left\langle TXT^{-1} \right\rangle = T^C \langle X \rangle; \qquad \left\langle TPT^{-1} \right\rangle = T^C \langle P \rangle. \tag{2}$$

Then (and only then) can one follow Callender's argument to conclude that the quantum T transforms X and P like their classical analogues. Namely, assume the standard classical transformation rules: $T^C \langle X \rangle = \langle X \rangle$ and $T^C \langle P \rangle = -\langle P \rangle$. Then Callender's correspondence rule (2) is true if and only if:

$$\begin{aligned} \left\langle TPT^{-1} \right\rangle &= T^C \langle P \rangle \\ &= -\langle P \rangle = \langle -P \rangle . \end{aligned}$$

Since this holds for any initial state $\psi \in \mathcal{H}$, it follows then (and only then)[10] that $TPT^{-1} = -P$. And, by the obvious symmetric argument, we also have that $TXT^{-1} = X$.

Notably, adopting the correspondence rule (2) encodes an assumption that T must on average behave like its classical analogue T^C. Ehrenfest's theorem does not guarantee this. It only guarantees a relationship between the expectation values of X and P,

[10] Here we make use of Theorem I of (Messiah 1999, §XV.1): A necessary and sufficient condition for two linear operators A and B to be equal is that $\langle \psi, A\psi \rangle = \langle \psi, B\psi \rangle$ for any state $\psi \in \mathcal{H}$.

namely that these quantities satisfy the Hamiltonian equations. So, it is the implicit correspondence rule that guarantees the quantum T will transform X and P like their classical analogues, not Ehrenfest's theorem.

This fact can also be seen explicitly, by observing that Ehrenfest's theorem is equally compatible with non-standard correspondence rules, such as:

$$\left\langle TXT^{-1}\right\rangle = T^C\left\langle X\right\rangle; \qquad -\left\langle TPT^{-1}\right\rangle = T^C\left\langle P\right\rangle, \tag{3}$$

from which it would follow that $\langle TPT^{-1}\rangle = -T^C\langle P\rangle = \langle P\rangle$. If this correspondence rule were correct, then it would provide an argument that T is a unitary operator, rather than an antiunitary one[11]. And if we were to adopt this time reversal operator, then as David Albert has suggested, ordinary non-relativistic quantum mechanics "is *not* invariant under time reversal" (Albert 2000, p.14). Indeed, perhaps an unusual correspondence rule such as (3) is one way to explicate Albert's view in response to Callender.

So, what justifies the choice of Callender's correspondence rule over some alternative? If we answer, 'because quantum physics should be like classical physics in this respect,' then we have failed to characterize time reversal without appeal to classical physics. Moreover, we have still not provided a characterization that applies to non-classical observables like spin. Thus, while Callender's approach does provide a very interesting account of the meaning of T, it does not fully resolve the problem the problem of conventionality introduced above.

4.2. **Albert's stack of pancakes.** David Albert (2000) has responded to the problems with the received view of time reversal by suggesting it be completely revised. Albert begins by proposing a very general characterization of time reversal. This project is indeed very much in the spirit of the one proposed above, in which we seek to recover the meaning of time reversal from independent principles. However, I will argue that Albert's characterization of time reversal is too weak. Although he seems to take his account to preclude the possibility that time reversal in quantum theory be antiunitary, I argue that the account that he has provided doesn't settle the matter either way. Nevertheless, I believe that we can learn something by working out Albert's view in somewhat more detail.

In a given reference frame, Albert suggests that the history of the universe is like a tall stack of pancakes: each slice represents an instantaneous state of the universe. Albert's conception of the time reversal transformation is then (roughly) that it flips the stack of pancakes. Albert writes:

[11]Namely: T could not be antiunitary given this correspondence rule, since (3) implies that $T[X,P]T^{-1} = [X,P]$, and hence that $Ti\hbar T^{-1} = i\hbar$. Assuming that transition probabilities are preserved by T, the corollary to Wigner's theorem thus implies that this T is unitary.

> Suppose that the true and complete fundamental physical theory of the world is something called T. Then any physical process is necessarily just some infinite sequence $S_I \dots S_F$ of instantaneous *states* of T. And what it is for that process to happen *backward* is just for the sequence $S_F \dots S_I$ to occur.

In the case of quantum theory, Albert seems to assume that 'flipping the stack' could not possibly involve complex conjugation. This kind of thinking seems to have lead Albert to think that time reversal is not antiunitarity, and hence that "the dynamical laws that govern the evolutions of quantum states in time cannot possibly be invariant under *time reversal*" (Albert 2000, 132). But why *couldn't* flipping the stack of pancakes involve conjugation? Albert's account doesn't settle the matter one way or the other. Indeed, consideration of a concrete example suggests that time reversal could (and, as I'll argue later, should) involve conjugation.

Consider a free wave packet propagating along the x spatial dimension. Its wavefunction can be expressed as a linear combination of plane waves:

$$\Psi(x,t) = \int_{-\infty}^{\infty} f(\mu) e^{i(\mu x - \xi t)} d\mu. \tag{4}$$

Here, μ is the momentum and ξ is the energy of each component plane wave. Let's focus our attention on just one of these plane waves, with wavefunction,

$$\psi(x,t) = e^{i(\mu x - \xi t)}.$$

Since the exponential is just a complex number on the unit circle, we can visualize this

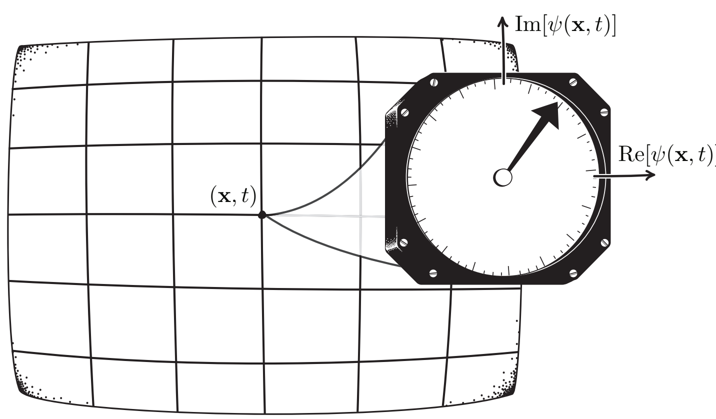

Figure 1: Visualizing a plane wave.

wave function as an arrow on a dial (Figure 1), where at a given spacetime point (x,t), the arrow makes the angle $\theta = (\mu x - \xi t)$ with the horizontal. If we fix x and let t increase, and if the energy ξ of the plane wave is positive, then the arrow will spin smoothly around the face of the dial in the *clockwise* direction.

Albert suggests that time reversal has the effect of arranging the states described by $\psi(x,t)$ in the opposite order. What kind of wavefunction describes such a reordering in our plane wave system? One naïve answer would be to write down

$$\hat{\psi}(x,t) = e^{i(\mu x + \xi t)} = e^{i(\mu x - \hat{\xi} t)},$$

where $\hat{\xi} = -\xi$. Fixing x and letting t increase in this wavefunction can be visualized as an arrow spinning *counter-clockwise* around the face of the dial. But strangely, although the resulting wavefunction $\hat{\psi}(x,t)$ has the same momentum μ as the original, it has opposite energy: $\hat{\xi} = -\xi$. The natural implementation of Albert's notion of time reversal thus takes positive-energy plane waves to negative-energy plane waves, and vice versa.

Such a characterization of time reversal may appear unfortunate. Why should an order-reversal change the sign of energy? Perhaps we have merely failed to correctly 'flip the stack.' So, consider another 'order-reversing' transformation that avoids the problem of negative energy: we can simply apply complex conjugation as well. Conjugating $\hat{\psi}(x,t)$ produces the wavefunction:

$$\hat{\psi}^*(x,t) = \left(e^{i(\mu x + \xi t)}\right)^* = e^{-i(\mu x + \xi t)} = e^{i(\hat{\mu} x - \xi t)},$$

where $\hat{\mu} = -\mu$. This wavefunction describes a plane wave with the same (positive) energy as the original, but opposite momentum μ. Indeed, this is just the standard way of time-reversing a plane wave.

So, does flipping the stack reverse the sign of a plane wave's momentum, or of its energy? Equivalently, does it involve conjugation, or not? Albert's account alone does not seem to provide an answer. However, our examination of the plane wave reveals that one's beliefs about energy may be relevant; in particular, if the energy of a time-reversed plane wave is positive, then it seems time reversal *must* involve conjugation. Indeed, a related condition will appear in the account of time reversal that I will propose below. But first, let us recall the account proposed by Wigner that inspired it.

4.3. **Wigner's promising alternative.** Eugene Wigner (1931, §26) gave an alternative derivation of antiunitarity, which seems to have been forgotten in most modern treatments[12]. This approach does not rely on any classical correspondence rule. Instead, Wigner assumes that T-reversal invariance is a *necessary condition* on T, whatever its precise nature may be. He then uses this condition to derive antiunitarity. Unfortunately, Wigner's argument still suffers from one of the misfortunes of the received view: it assumes time reversal invariance, instead of deriving it, and fails to accommodate the fact

[12]An exception is (Sakurai 1994).

that some interactions *may not be* time reversal invariant. Moreover, Wigner still does not provide a derivation of way time reversal transforms observables. However, we will see in the next section that a simple refinement of his argument is enough to overcome all of these problems.

Wigner's assumption of time reversal invariance may not be clearly recognizable as such. In what follows, I briefly review what Wigner says, and then show that it is equivalent to the common notion of time reversal invariance. The latter can be characterized by the following.

Definition 1. Quantum theory is T-reversal invariant if: (1) T preserves transition probabilities, in that $|\langle T\psi, T\phi\rangle| = |\langle\psi, \phi\rangle|$ for all $\psi, \phi \in \mathcal{H}$; and (2) if $\psi(t)$ is a trajectory satisfying the equations of motion, then so is $T\psi(-t)$.

Wigner begins by writing down his assumption that $|\langle T\psi, T\phi\rangle| = |\langle\psi, \phi\rangle|$; this satisfies the first part of the Definition 1. But in place of the second part, Wigner wrote:

> The following four operations, carried out in succession on an arbitrary state, will result in the system returning to its original state. The first operation is time inversion, the second time displacement by t, the third again time inversion, and the last on again time displacement by t. (Wigner 1931, 326)

To see why this is equivalent to the second part of Definition 1, let us represent Wigner's sequence of transformations in terms of the transformations illustrated in Figure 2. Imagine we roll our toy car forward, flip it, roll it back again, and finally flip it back.

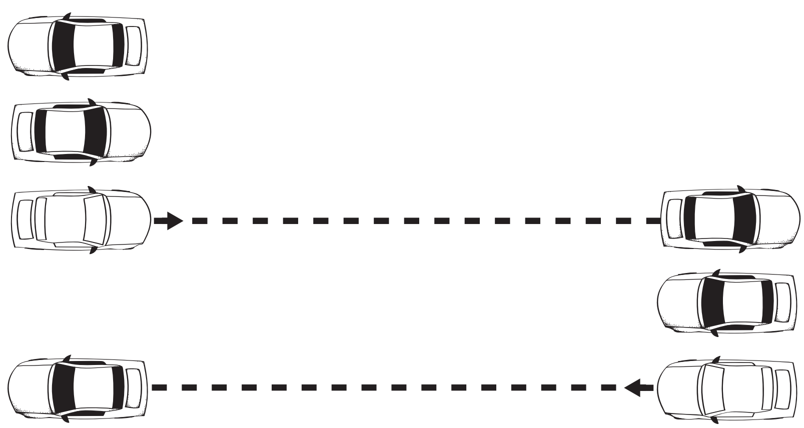

Figure 2: Wigner's sequence of transformations: reverse, evolve for time t, reverse, and evolve for time t again.

Then the claim is that the initial state of the car is the same as its final state.

The consequences of this assumption for quantum mechanics are clear when we write it down in terms of operators. First, the 'time displaced' state $\psi(t)$ of an initial state $\psi(0) = \psi_0$ is characterized by the application of a unitary propagator $U_t = e^{-itH}$, where H is the Hamiltonian[13]; in particular,

$$\psi(t) = e^{-itH}\psi_0.$$

Second, denote Wigner's 'time inversion' by an operator T. In these terms, Wigner's assumption says that:

$$e^{-itH}T^{-1}e^{-itH}T\psi_0 = \psi_0. \tag{5}$$

Multiplying on the left by Te^{itH}, we can see that this condition is true if and only if

$$\begin{aligned} e^{-itH}T\psi_0 &= Te^{itH}\psi_0 \\ &= T\psi(-t). \end{aligned} \tag{6}$$

In other words, if $\psi(t)$ is a possible trajectory, then so is $T\psi(-t)$. This is just the second condition of Definition 1. Thus, whatever a sufficient characterization of T might be, *Wigner took it as a necessary condition that quantum theory be T-reversal invariant.*

The first upshot is that, unlike the received view, Wigner has provided a general context in which it is true that time reversal preserves transition probabilities: the context in which quantum theory is time reversal invariant. The second is that, *assuming* that quantum theory is time reversal invariant, one can now provide an argument that T is antiunitary, without appeal to classical transformation rules. Here is a sketch of how this argument goes.

Wigner thought that T does not change the energy of stationary states[14]. This immediately implies that T commutes with the Hamiltonian[15]. We now observe (for reductio) that if T were unitary, it would commute with the evolution operator e^{-itH}, and hence by Equation (6),

$$\begin{aligned} Te^{-itH}\psi_0 &= e^{-itH}T\psi_0 \\ &= T\psi(-t) \\ &= Te^{itH}, \end{aligned}$$

which is a contradiction. Moreover, since Wigner assumes time reversal invariance from the outset, he can infer that time reversal preserves transition probabilities. Hence, the premise of Wigner's theorem is satisfied, and it follows that T is antiunitary.

[13]Here and throughout we adopt units in which $\hbar = 1$.

[14]He did not justify this assumption, but my refinement of his argument in the next section will do so.

[15]*Proof:* Let ψ be a stationary state, where $H\psi = e\psi$ and T is energy-preserving: $HT\psi = eT\psi$. Then $HT\psi = Te\psi = TH\psi$, and $[T, H] = 0$.

Unfortunately, the difficulties with Wigner's argument are the familiar ones: even on the standard definition of T, quantum theory is *not* generally time reversal invariant. Thus, like the received view, Wigner's own application of Wigner's theorem is apparently unjustified. Moreover, although no appeal to classical mechanics has been made, Wigner still hasn't provided a rigorous derivation of the way time reversal transforms observables. Indeed, Wigner (1931) proceeds to simply posit the classical transformation rules without argument. To complete Wigner's argument, one would now like a *derivation* of the transformation rules of observables like X, P and S.

In the next section, we will seek to meet these challenges.

5. A FOUNDATIONAL APPROACH

My purpose so far has been to illustrate two open problems: the derivation of the antiunitarity of time reversal without appeal to classical physics, and the derivation of the transformation rules for time reversal for both classical and non-classical observables. As we will now see, a refinement of Wigner's approach does lead to a resolution of these problems. I call this strategy the 'foundational approach.'

I will follow the standard practice of taking the time reversal *operator* T to be a Hilbert space bijection on the vectors in $\mathcal{H}$, and the time reversal *transformation* to be a mapping on the dynamical trajectories $\psi(t)$ in $U_t\mathcal{H}$. Our strategy will then be to adopt four plausible conditions about this transformation, and show they essentially determine the meaning of time reversal. To begin, let us observe a simple replacement for Wigner's overzealous assumption that quantum theory is in general T-reversal invariant.

Condition 1. *Free motion T-invariance.* Let Γ be a collection of free Hamiltonians. Then the restriction of quantum theory to Γ is T-reversal invariant; namely:

(1) $|\langle T\psi, T\phi\rangle| = |\langle\psi, \phi\rangle|$ for all $\psi, \phi \in \mathcal{H}$; and
(2) If $\psi(t)$ is a solution to Schrödinger's equation, then so is $T\psi(-t)$.

Instead of following Wigner in demanding time reversal invariance *for all* interactions, we assume it only in the special case of 'free' motion, i.e., motion represented by a Hamiltonian incorporating no potentials, collisions, or any other interactions – such as $H_0 = P^2/2m$. Happily, this weaker assumption seems to enjoy the possibility of being *true*, as far as we can tell[16]. It also respects the demand of Albert, Artzenius and Greaves

[16] It need not be. The existence of a spherically symmetric, non-degenerate system with a permanent electric dipole moment would be T-violating. However, at the time of this writing, no such system has ever been discovered. For an overview, see (Khriplovich and Lamoreaux 1997).

discussed above: we do not assume that quantum theory is *generally* time reversal invariant; on our view, this remains a question of experiment. Our assumption is merely that time reversal invariance holds in the special case of free motion.

Condition 2. *Non-negative spectrum.* The spectrum of a free Hamiltonian H_0 is non-negative.

Note that our condition does not prohibit negative energy, except in the absence of interactions. This condition trivially holds of ordinary quantum systems. It can also be seen as a consequence of a more general characterization of free Hamiltonians[17]. Here, we assume it as a postulate, in place of Wigner's less-obvious assumption about stationary states.

Condition 3. *Involution.* In a given superselection sector, $T^2 = e^{i\theta}$ for some $\theta \in \mathbb{R}$.

This assumption takes seriously the nomenclature: time reversal is a *reversal*. As such, applying it twice brings us back to where we started up to phase; such transformations are called *involutions*. Notably, the phase may differ among distinct superselection sectors, in particular the fermion and boson sectors.

Condition 4. *Spatial Isotropy*. If R^α_θ is any rotation operator through an angle θ about an axis α, then $[T, R^\alpha_\theta] = 0$.

This encodes the assumption that there is no preferred spatial direction. As a consequence, the effect that time reversal has on a state should not depend on whether or not an arbitrary spatial rotation has been applied to that state.

We are now in a position to provide solutions to the problems posed above. We begin with an argument that time reversal must be implemented by an antiunitary operator T.

Proposition 1. *Suppose Conditions 1 and 2 are true. Then T is antiunitary.*

This proposition can be viewed as a precise refinement of the argument sketched by Wigner. The result is neat foundational solution to Problem 1 posed above: we show that T is antiunitarity, and do so on quantum theory's own terms, with no appeal whatsoever to classical physics.

Wigner himself did not go on to derive the way that time reversal transforms observables like position, momentum, and angular momentum. Indeed, as it turns out, Conditions 1 and 2 are not enough to derive them uniquely. However, a modest addition

[17]For example, that of (Zeidler 2009, 526).

to these assumptions *is* enough to derive the transformation rules; namely, we introduce Conditions 3 and 4, that T be a spatially isotropic involution.

To begin, let us understand observables like X, P or S in terms of the role they play in the Galilei group. In order to avoid the domain problems of unbounded operators[18], we express the relation between position (X) and momentum (P) in terms of the one-parameter groups of boosts ($U_a = e^{iaX}$) and spatial translations ($V_b = e^{ibP}$) that they respectively generate[19]; namely,

$$U_a V_b = e^{iab} V_b U_a.$$

Similarly, the angular momentum operators S^α (for either spin and orbital degrees of freedom) generate rotation operators R^α_θ about the α axis. They are related to U_a and V_b by the usual rotation matrices:

$$\begin{aligned}
R^z_\theta U^x_a R^z_{-\theta} &= U^x_{a\cos\theta} U^y_{-a\sin\theta} & \qquad R^z_\theta V^x_b R^z_{-\theta} &= V^x_{b\cos\theta} V^y_{-b\sin\theta} \\
R^z_\theta U^y_a R^z_{-\theta} &= U^x_{a\sin\theta} U^y_{a\cos\theta} & \qquad R^z_\theta V^y_b R^z_{-\theta} &= V^x_{b\sin\theta} V^y_{b\cos\theta} \\
R^z_\theta U^z_a R^z_{-\theta} &= U^z_a & \qquad R^z_\theta V^z_a R^z_{-\theta} &= V^z_a,
\end{aligned}$$

and so on for rotations about the other axes.

We finally note that in the presence of superselection rules, the unitary operators U_a and V_b will be taken to act invariantly on the individual superselection sectors[20]. This, together with Condition 3, implies that $[T^2, U_a] = [T^2, V_b] = 0$. For our purposes, we take this behavior to be a brute fact about physical representations of the Galilei group; however, it can be justified either by appeal to the explicit form of U_a and V_b, or by the empirical fact that nature does not admit coherent superpositions from across superselection sectors[21].

With this apparatus in place, the foundational approach to time reversal allows us to derive the transformation rules for such observables on the basis of these relations. First, we have the following.

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[18] See (Bogolubov, Logunov, Oksak, and Todorov 1990, §6) for a discussion.

[19] In fact, X must be rescaled by a constant factor m in order to generate the boosts; we set this factor to 1 without loss of generality.

[20] For an overview of superselection rules, see (Earman 2008).

[21] Indeed, it is a simple matter to show that if (i) $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ is a partition of $\mathcal{H}$ into its distinct superselection sectors; (ii) every linear combination ψ of vectors from both $\mathcal{H}^+$ and $\mathcal{H}^-$ is a mixed state; and (iii) $T^2\phi^+ = e^{i\theta}\phi^+$ for $\phi^+ \in \mathcal{H}^+$ and $T^2\phi^- = e^{i\lambda}\phi^-$ for $\phi^- \in \mathcal{H}^-$ (Condition 3), then $[T^2, A] = 0$ for all linear operators A.

Proposition 2. *Suppose Conditions 1 - 4 are true. If T is in a representation of the inhomogeneous Galilei group, and if U_a, V_b are continuous subgroups of boosts and spatial translations along some axis, then $TU_aT^{-1} = U_a^{\mp 1}$ and $TV_bT^{-1} = V_b^{\pm 1}$.*

The proposition reveals that there are really *two* time reversal operators in quantum mechanics. Recall that $U_a = e^{iaX}$ and $V_b = e^{ibP}$, and thus $TU_aT^{-1} = e^{-iaTXT^{-1}}$ and $TV_bT^{-1} = e^{-ibTPT^{-1}}$. We now see that the first time reversal operator ($U_a \mapsto U_a^{-1}$ and $V_b \mapsto V_b$) has the form,

$$X \mapsto X$$
$$P \mapsto -P$$

which matches the standard transformation rule of the received view. The second ($U_a \mapsto U_a$ and $V_b \mapsto V_b^{-1}$) has the form,

$$X \mapsto -X$$
$$P \mapsto P$$

which matches the standard *space and time* reversal operator[22]. Both are reasonable time reversal operators on this account. Indeed, we wouldn't expect to distinguish between them given only assumptions that are symmetric with respect to X and P.

Still, one might worry: the received view is that there is a *unique* time reversal operator. Isn't it a shortcoming that the foundational approach recovers two instead of one? I claim that it isn't. From a foundational perspective, there actually *are* many time reversal operators: one preserves spatial positions, and the other reverses them. The interesting fact is not which of them is 'correct,' but rather that both reverse position in the the opposite way that they reverse momentum. We now have an explanation of these widely-accepted conventions in terms of a few foundational principles.

On the other hand, there is no such question in the case of angular momentum. It is a much simpler matter to show that, as a consequence of the isotropy condition, the time reversal transformation rules in this case are uniquely determined.

Proposition 3. *Suppose Conditions 1, 2 and 4 are true. If S^α is the generator of rotations about an axis α (either in orbital angular momentum or spin), then $TS^\alpha T^{-1} = -S^\alpha$.*

Thus, the foundational approach provides a complete solution to Problem 2: we have a derivation of the way time reversal transforms observables in quantum mechanics, including those that have no classical analogue (like spin). Recall that the standard view only suggested that spin reverses sign because it is 'like' angular momentum in classical

[22]Intuitively: If both space (x) and time (t) are sent to their negatives, then velocities (dx/dt) remain fixed.

mechanics. The foundational approach argument completely avoids such analogies. In their place, we make use only of the group structure of the Galilei group, together with the spatial isotropy afforded by Condition 4.

6. How general is the foundational approach?

We have conceived of the time reversal transformation as a transformation on set of dynamical trajectories $\psi(t)$ of quantum theory. The foundational approach was then to require certain necessary conditions on this transformation, and then seek to derive its precise meaning. It should be emphasized that this approach can only get off the ground once we've specified a dynamical theory. For quantum theories that advocate modifications or additions to the standard dynamics, our account may say different things. In particular, if a stochastic parameter is introduced, the foundational approach is unlikely to be applicable. On the other hand, in dynamical theories that depend only on a single time parameter t, the foundational approach can be shown to give rise to results very similar to the ones produced above. We illustrate this below with a discussion of dynamical collapse theories, followed by applications of the foundational approach to both Bohmian mechanics and classical Hamiltonian mechanics.

6.1. **Dynamical collapse.** Dynamical collapse theories are a null case for the foundational approach to time reversal: our account says almost nothing about their symmetries. One of the essential characterizations of quantum theory that our account makes use of is its unitary dynamics. In particular, a unitary propagator evolves a quantum system with respect to a single time parameter t. In contrast, dynamical collapse theories introduce a stochastic parameter[23] into the dynamics, and thus don't have single-parameter trajectories. In such cases, the extra stochastic parameter is an essential part of the dynamics, setting these theories outside the scope of our discussion of dynamical symmetries.

This is not so unusual. Typical discussions of dynamical symmetries are about the way fixed trajectories relate to other fixed trajectories. This makes perfect sense for deterministic theories, which constrain the behavior of particular trajectories. But stochastic theories normally make claims about an *ensemble* of trajectories. So, whatever a 'dynamical symmetry' is typically taken to mean, must be something very different in a stochastic theory, whether it be GRW, CSL, or ordinary statistical mechanics.

6.2. **Bohmian mechanics.** In Bohmian mechanics, our foundational approach can be applied, and it leads to a proposition much like the ones we have seen above. However,

[23]This parameter functions as an instantaneous Gaussian 'hit' in the GRW picture (Ghirardi, Rimini, and Weber 1986), and as a Markov process of continuous evolution in the CSL picture (Pearle 1989).

the two dynamical equations that appear in this theory call for a slightly more subtle discussion.

According to Bohmian mechanics, the real stuff of the world is governed by the *guidance equation*. For a single particle, the guidance equation takes the form:

$$\frac{dx}{dt} = -\frac{1}{m}\mathrm{Im}\frac{\nabla\psi(x)}{\psi(x)}, \tag{7}$$

where $\psi(x) = \langle\varphi_x, \psi\rangle$ is the position wavefunction for ψ. The behavior of the wavefunction is in turn determined by the ordinary Schrödinger equation,

$$i\frac{\partial}{\partial t}\psi = H\psi. \tag{8}$$

Although Bohmian mechanics employs two distinct dynamical equations, the guidance equation is the one thought to govern the stuff of the world (often refered to as elements of the *primitive ontology*.). As a consequence, the notion of time reversal most relevant for the Bohmian is the one that transforms solutions to the guidance equation (7), and not solutions to the Schrödinger equation (8). That is, the Bohmian is interested in the meaning of the 'Bohmian' time reversal operator.

Definition 2. The *Bohmian time reversal operator* is a bijection $T : \mathbb{R}^{3n} \to \mathbb{R}^{3n}$ on Bohmian configuration space.

There is now a subtlety for us to deal with. While the Bohmian time reversal operator acts on the *positions* of Bohmian particles, the guidance equation (7) depends also on a *wavefunction* ψ, which lives in abstract Hilbert space $\mathcal{H}$. How are we to determine what time reversal does to this term, while still demanding that T act fundamentally on the positions living in Bohmian configuration space?

I propose one simple way to make the connection between Hilbert space and configuration space, by allowing the position representation to have 'special' status in the characterization of time reversal. To begin, let us follow the typical practice of defining a Hilbert space representation in terms of an infinite dimensional function space, but taking configuration space as the domain of these functions. That is, we let $\mathcal{H}$ contain the square-integrable functions from Bohmian configuration to the complex numbers, $f : \mathbb{R}^{3n} \to \mathbb{C}$. The Bohmian time reversal operator on $\mathbb{R}^{3n}$ then induces a canonical transformation on $\mathcal{H}$, which we can state as an explicit part of our picture of Bohmian mechanics.

Definition 3. The *action of Bohmian time reversal on Hilbert space* is given by an operator $\hat{T} : \mathcal{H} \to \mathcal{H}$ such that, if T is the Bohmian time reversal operator, then $\hat{T}\psi(x) = \psi(Tx)$ for all $\psi \in \mathcal{H}$.

We are now nearly ready to provide a foundational characterization of the meaning of Bohmian time reversal. However, I would first like to introduce one further condition on the Bohmian T, which is required for the characterization that I will give below. The idea is that in Bohmian mechanics, a symmetry operator should give special treatment to those 'vectors' that form the position basis[24]. In particular, position basis vectors should not be transformed to non-basis vectors under time reversal, if these vectors are to maintain their status as characterizing a Bohmian particle at a point. This just a reflection of the Bohmian perspective that position is the 'preferred' representation, in directly characterizing a system's primitive ontology. We state this as a necessary (but not sufficient) condition on the Bohmian time reversal operator.

Condition 5. The Bohmian time reversal operator T is such that, if φ_{x_0} is a position eigenfunction, then so is its image under the action of T; i.e., $\hat{T}\varphi_{x_0} = \varphi_{x_1}$ for some x_1.

Notably, this definition does not say whether or not $\hat{T}$ is linear or antilinear. Familiarity with the above case of ordinary quantum theory may lead one to specify that $\hat{T}$ is antilinear. However, *this specification just isn't required* for a characterization of the meaning of T, as the following proposition illustrates.

Proposition 4. *Suppose that T is an involution ($T^2 = I$) on configuration space, that the action of T on $\mathcal{H}$ satisfies Condition 5, and that the Bohmian guidance equation is T-reversal invariant for the free particle Hamiltonian. Then the action of T on $\mathcal{H}$ must be antilinear, and either $Tx = x$, or else $Tx = x_0 - x$ for some fixed x_0.*

This result is an exact analogue of our characterization of time reversal in ordinary quantum mechanics; however, it is less visible because T does not act directly on Bohmian velocities. To make this explicit, let's observe how the above two time reversal operators act on Bohmian velocities. In general, the time reversal transformation takes a Bohmian trajectory $x(t)$ to $Tx(-t)$. But using the chain rule, it's easy to see that,

$$\frac{dTx(-t)}{dt} = \frac{d(-t)}{dt}\frac{dTx(-t)}{d(-t)} = -\frac{dTx(-t)}{d(-t)} = -\frac{dTx(t)}{dt}.$$

[24]As is well known, the position basis 'vectors' $\varphi_{x_0} = \delta(x - x_0)$ are not square integrable functions. To include such vectors while mainting rigor, $\mathcal{H}$ should properly be considered a 'rigged' Hilbert space, namely, one that has been expanded to include a complete set of eigenfunctions for unbounded self-adjoint operators like X and P. See (Melsheimer 1974), (Ballentine 1998, §1.3-1.4), and the references therein for an introduction.

So, Proposition 4 entails that time reversal has one of two effects on Bohmian positions and velocities. Either

$$x \mapsto x$$
$$\frac{dx}{dt} \mapsto -\frac{dTx}{dt} = -\frac{dx}{dt},$$

as is the case on the standard definition of time reversal; or else

$$x \mapsto x_0 - x$$
$$\frac{dx}{dt} \mapsto -\frac{dTx}{dt} = \frac{dx}{dt}$$

as is the case on the standard definition of space-and-time reversal, where the constant x_0 represents our freedom to choose an axis about which to reverse space.

The advantage of this approach to Bohmian time reversal is that it does not presuppose any facts about time reversal in ordinary quantum mechanics. The meaning of T is necessitated by the weakening of Wigner's T-reversal invariance assumption, the fact that T is an involution, and the nature of the primitive ontology in Bohmian mechanics.

It is often assumed that time reversal has a particular meaning in Bohmian mechanics[25]. In place of this unjustified assumption, we now have an *argument* for why the Bohmian time reversal operator means what it does. However, this argument does presuppose the standard Bohmian guidance equation. Since there are many empirically adequate guidance equations, as Stone (1994) has pointed out, this assumption has to be justified. In particular, Dürr, Goldstein, and Zanghí (1992) have argued that there is a unique guidance equation; unfortunately, they adopt the standard meaning of time reversal as a premise. That argument is therefore not available to us, on pain of circular reasoning. On our approach, Bohmians must rely on some other means of establishing the guidance equation, such as that suggested by Bohm (1952).

6.3. **Classical mechanics.** The received view of time reversal assumes a correspondence rule between quantum and classical mechanics. On that view, we need a completely different technique if we wish to establish the meaning of time reversal in classical mechanics itself. On the other hand, our foundational approach to time reversal *can* be applied to classical mechanics, just as it applies to quantum mechanics and Bohmian mechanics. This result is characterized by the following.

Proposition 5. *Suppose that T is a linear bijection on phase space, that T is an involution ($T^2 = I$), and that the Hamiltonian equations are T-reversal invariant for the free*

[25]For example, see (Dürr, Goldstein, and Zanghí 1992), (Deotto and Ghirardi 1998) (Allori, Goldstein, Tumulka, and Zanghi 2008).

particle Hamiltonian. Then either $Tp = -p$ *and* $Tq = q$*, or else* $Tp = p$ *and* $Tq = q_0 - q$ *for some fixed* q_0.

As above, this result actually recovers two time reversal operators. The first ($Tq = q$ and $Tp = -p$) is the standard time reversal operator. The second option ($Tq = q_0 - q$ and $Tp = p$) is the standard *space and time* reversal operator, where the constant q_0 represents our freedom to choose an axis about which to flip space.

The foundational approach to time reversal thus turns out to be surprisingly general: it solves the problems of time reversal not only in quantum mechanics and Bohmian mechanics, but in classical mechanics too.

7. Conclusion

The meaning of time reversal is often passed over so quickly that it appears conventional. This difficulty is compounded by the problems associated with the received justification of the meaning of time reversal. Although Craig Callender and David Albert have both made progress in attempting to repair or reject the received view (respectively), neither has produced an solution to these problems that adequately establishes the meaning of time reversal in quantum mechanics.

However, as we've now seen, Wigner's account of time reversal contains the seeds of a promising alternative. A simple refinement of his assumptions led us to a more plausible proof of antiunitarity in Proposition 1, which makes no appeal to classical physics. A modest addition to his assumptions allowed us to derive the way time reversal transforms observables in Propositions 2 and 3, including observables that have no classical analogue. Finally, we found that this approach to time reversal applies even beyond quantum theory, but allows for a derivation of the transformation rules for time reversal in Bohmian mechanics (Proposition 4) and classical Hamiltonian mechanics (Proposition 5). Indeed, it's plausible that this approach might also illuminate the meaning of time reversal in other theories as well. However, such an exploration must remain a topic for another paper.

Appendix: Proofs of propositions

Condition 1. *Free motion* T*-invariance.* Let Γ be a collection of free Hamiltonians. Then the restriction of quantum theory to Γ is T-reversal invariant; namely:

(1) $|\langle T\psi, T\phi\rangle| = |\langle \psi, \phi\rangle|$ for all $\psi, \phi \in \mathcal{H}$; and
(2) If $\psi(t)$ is a solution to Schrödinger's equation, then so is $T\psi(-t)$.

Condition 2. *Non-negative spectrum.* The spectrum of a free Hamiltonian H_0 is non-negative.

Condition 3. *Involution.* In a given superselection sector, $T^2 = e^{i\theta}$ for some $\theta \in \mathbb{R}$.

Condition 4. *Spatial Isotropy*. If R^α_θ is any rotation operator through an angle θ about an axis α, then $[T, R^\alpha_\theta] = 0$.

Condition 5. The Bohmian time reversal operator T is such that, if φ_{x_0} is a position eigenfunction, then so is its image under the action of T; i.e., $\hat{T}\varphi_{x_0} = \varphi_{x_1}$ for some x_1.

Proposition 1. *Suppose Conditions 1 and 2 are true. Then T is antiunitary.*

Proof. Let $\psi(t) = e^{-itH_0}\psi_0$ describe a dynamical trajectory, with initial state ψ_0 and free Hamiltonian H_0. Substituting $t \mapsto -t$, we derive an equivalent formulation:

$$\psi(-t) = e^{itH_0}\psi_0. \tag{9}$$

Now, Free Motion T-Invariance (Condition 1) implies that the trajectory $T\psi(-t)$ with initial state $T\psi_0$ also satisfies normal Schrödinger evolution:

$$T\psi(-t) = e^{-itH_0}T\psi_0. \tag{10}$$

But substituting Equation (9) into the LHS of Equation (10) gives:

$$Te^{itH_0}\psi_0 = e^{-itH_0}T\psi_0.$$

This equation holds for any initial state ψ_0 we might have selected, so the operators on the left and right can be equated, namely, $Te^{itH_0} = e^{-itH_0}T$. Therefore,

$$e^{-itH_0} = Te^{itH_0}T^{-1} = e^{TitH_0T^{-1}}.$$

It follows that $-iH_0 = TiH_0T^{-1}$. But T satisfies the premises of Wigner's theorem (Condition 1), and thus is either unitary or antiunitary. Moreover, T cannot be unitary. For if it were, then we could divide both sides by i to get that $TH_0T^{-1} = -H$. But this would imply that are negative elements in the spectrum of H_0. In particular, from the general fact that $\mathrm{sp}(AB) \cup \{0\} = \mathrm{sp}(BA) \cup \{0\}$ (Kadison and Ringrose 1983, Prop. 3.2.8), it follows that

$$\mathrm{sp}(-H) \cup \{0\} = \mathrm{sp}(THT^{-1}) \cup \{0\} = \mathrm{sp}(TT^{-1}H) \cup \{0\} = \mathrm{sp}(H) \cup \{0\},$$

i.e., H and $-H$ would have the same non-zero spectra. But by the spectral mapping theorem (Kadison and Ringrose 1983, Thm. 3.3.6), $\mathrm{sp}(-H) = \{-h \mid h \in \mathrm{sp}(H)\}$. So, for any positive element $h \in \mathrm{sp}(H)$, there is a negative element $-h \in \mathrm{sp}(-H)$, which is non-zero and therefore also in $\mathrm{sp}(H)$. This contradicts the non-negative spectrum assumption (Condition 2); thus, T must be antiunitary. □

The proof of our next proposition is streamlined by a lemma.

Lemma 1. *Let* $\mathbf{U}$ *and* $\mathbf{V}$ *be continuous subgroups of boosts and spatial translations, respectively, in a representation of the full inhomogeneous Galilei group* $\mathbb{G}_I$*. Then for any* $T \in \mathbb{G}_I$ *satisfying Condition 4, and for any* $U_a \in \mathbf{U}$*,* $V_b \in \mathbf{V}$*, there exists some* $\mu, \nu \in \mathbb{R}$ *such that* $TU_aT^{-1} = U_{\mu a}$ *and* $TV_bT^{-1} = V_{\nu b}$*.*

Proof. We begin by recalling that the boosts $\mathbf{U}$ and translations $\mathbf{V}$ are *normal subgroups* of the inhomogeneous Galilei group (Kim and Noz 1986, Ch.8 Problem 7); that is, $T\mathbf{U}T^{-1} = \mathbf{U}$ and $T\mathbf{V}T^{-1} = \mathbf{V}$ for any $T \in \mathbb{G}_I$. Three orthogonal elements U_a^x, U_a^y, U_a^z of $\mathbf{U}$ will thus be transformed to three arbitrary elements of $\mathbf{U}$ under T:

$$\begin{aligned} TU_a^xT^{-1} &= U^x_{\mu_1 a}U^y_{\mu_2 a}U^z_{\mu_3 a}, \\ TU_a^yT^{-1} &= U^x_{\mu_4 a}U^y_{\mu_5 a}U^z_{\mu_6 a}, \\ TU_a^zT^{-1} &= U^x_{\mu_7 a}U^y_{\mu_8 a}U^z_{\mu_9 a}. \end{aligned} \tag{11}$$

We now show that spatial isotropy (Condition 4) implies that all the off-diagonal μ-terms in (11) must vanish, while all the diagonal μ-terms must be equal.

Isotropy guarantees that $(TR_\theta^z)U_a^x(R^z_{-\theta}T^{-1}) = (R^z_\theta T)U^x_a(T^{-1}R^z_{-\theta})$. We may thus calculate the values of the left and right hand sides and then equate them. The LHS is

$$\begin{aligned} TR_\theta^zU_a^xR^z_{-\theta}T^{-1} &= T(U^x_{a\cos\theta}U^y_{-a\sin\theta})T^{-1} \\ &= (TU^x_{a\cos\theta}T^{-1})(TU^y_{-a\sin\theta}T^{-1}) \\ &= (U^x_{\mu_1 a\cos\theta}U^y_{\mu_2 a\cos\theta}U^z_{\mu_3 a\cos\theta})(U^x_{-\mu_4 a\sin\theta}U^y_{-\mu_5 a\sin\theta}U^z_{-\mu_6 a\sin\theta}) \end{aligned} \tag{12}$$

and the RHS is

$$\begin{aligned} R_\theta^zTU_a^xT^{-1}R^z_{-\theta} &= R^z_\theta(U^x_{\mu_1 a}U^y_{\mu_2 a}U^z_{\mu_3 a})R^z_{-\theta} \\ &= (R^z_\theta U^x_{\mu_1 a}R^z_{-\theta})(R^z_\theta U^y_{\mu_2 a}R^z_{-\theta})(R^z_\theta U^z_{\mu_3 a}R^z_{-\theta}) \\ &= (U^x_{\mu_1 a\cos\theta}U^y_{-\mu_1 a\sin\theta})(U^x_{\mu_2 a\sin\theta}U^y_{\mu_2 a\cos\theta})(U^z_{\mu_3 a}). \end{aligned} \tag{13}$$

Equating the x, y, and z terms in (12) and (13), we now have

$$\begin{cases} \mu_1\cos\theta - \mu_4\sin\theta = \mu_1\cos\theta + \mu_2\sin\theta & \Rightarrow \boxed{\mu_2 = -\mu_4} \\ \mu_2\cos\theta - \mu_5\sin\theta = -\mu_1\sin\theta + \mu_2\cos\theta & \Rightarrow \boxed{\mu_1 = \mu_5} \\ \mu_3\cos\theta - \mu_6\sin\theta = \mu_3 & \Rightarrow \boxed{\mu_3 = \mu_6 = 0} \end{cases}$$

where we have assumed without loss of generality that $a \neq 0$. Furthermore, by calculating $(TR^y_\theta)U^x_a(R^y_{-\theta}T^{-1}) = (R^y_\theta T)U^x_a(T^{-1}R^y_\theta)$, one can derive in just the same way that

$$\begin{cases} \mu_1\cos\theta + \mu_7\sin\theta = \mu_1\cos\theta - \mu_3\sin\theta & \Rightarrow \boxed{\mu_7 = -\mu_3} \\ \mu_2\cos\theta + \mu_8\sin\theta = \mu_2 & \Rightarrow \boxed{\mu_2 = \mu_8 = 0} \\ \mu_3\cos\theta + \mu_9\sin\theta = \mu_1\sin\theta + \mu_3\cos\theta & \Rightarrow \boxed{\mu_1 = \mu_9} \end{cases}$$

Combining these results, we find that the diagonal terms $\mu_1 = \mu_5 = \mu_9$ are all equal, while the other terms all vanish. Since an exactly similar argument holds for the subgroup of spatial translations $\mathbf{V}$, this proves the lemma. □

Proposition 2. *Suppose Conditions 1 - 4 are true. If T is in a representation of the inhomogeneous Galilei group, and if U_a, V_b are continuous subgroups of boosts and spatial translations along some axis, then $TU_aT^{-1} = U_a^{\mp 1}$ and $TV_bT^{-1} = V_b^{\pm 1}$.*

Proof. The previous lemma established that $TU_aT^{-1} = U_{\mu a}$ and $TV_bT^{-1} = V_{\nu b}$. Moreover, since T is an involution in a given superselection sector (Condition 3), and since U_a and V_b act invariantly on superselection sectors, we know that $T^2(U_a)T^{-2} = e^{i\theta}(U_a)e^{-i\theta} = U_a$. Combining these results, we immediately have that

$$U_a = T^2(U_a)T^{-2} = T(U_{\mu a})T^{-1} = U_{\mu^2 a},$$

and hence that $\mu = \pm 1$. Applying the same argument to V_b, it follows that $\nu = \pm 1$ as well. So, we now just need to establish that μ and ν have opposite signs.

To this end, let us again write $TU_aT^{-1} = U_{\mu a}$ and $TV_bT^{-1} = V_{\nu b}$. Since we found our first Proposition that T is antiunitary, $Te^{iab}T^{-1} = e^{-iab}$. Thus, when we apply T to both sides of the commutation relation

$$U_aV_b = e^{iab}V_bU_a,$$

we find that

$$\begin{aligned} & T(U_aV_b)T^{-1} = e^{-iab}T(V_bU_a)T^{-1} \\ \Rightarrow\ & (TU_aT^{-1})(TV_bT^{-1}) = e^{-iab}(TV_bT^{-1})(TU_aT^{-1}) \\ \Rightarrow\ & U_{\mu a}V_{\nu b} = e^{-iab}V_{\nu b}U_{\mu a} \\ \Rightarrow\ & e^{i\mu\nu ab}V_{\nu b}U_{\mu a} = e^{-iab}V_{\nu b}U_{\mu a}. \end{aligned}$$

Canceling $V_{\nu b}U_{\mu a}$ from both sides, we now have $e^{i\mu\nu ab} = e^{-iab}$, and hence $\mu\nu = -1+2\pi k$ for some integer k. But since μ and ν can only be ± 1, it follows that $k = 0$ and $\mu = -\nu$. Therefore, $TU_aT^{-1} = U_{\mp a} = U_a^{\mp 1}$ and $TV_bT^{-1} = V_{\pm b} = V_b^{\pm 1}$, and we are done. □

Proposition 3. *Suppose Conditions 1, 2 and 4 are true. If S^α is the generator of rotations about an axis α (either in orbital angular momentum or spin), then $TS^\alpha T^{-1} = -S^\alpha$.*

Proof. Since the S_α satisfy the angular momentum commutation relations, each can be expressed as the generator of a rotation operator, $R_\theta^\alpha = e^{i\theta S^\alpha}$. Thus,

$$TR_\theta^\alpha T^{-1} = Te^{i\theta S^\alpha}T^{-1} = e^{Ti\theta S^\alpha T^{-1}}.$$

But Condition 4 requires that $TR^{\alpha}_{\theta}T^{-1} = R^{\alpha}_{\theta} = e^{i\theta S^{\alpha}}$, so

$$e^{Ti\theta S^{\alpha}T^{-1}} = e^{i\theta S^{\alpha}},$$

and hence $TiS^{\alpha}T^{-1} = iS^{\alpha}$. Finally, our first proposition guarantees that T is antiunitary, so $TiS^{\alpha} = -iTS^{\alpha}$, and we have that

$$TS^{\alpha}T^{-1} = -S^{\alpha}.$$

□

Proposition 4. *Suppose that T is an involution ($T^2 = I$) on configuration space, that the action of T on $\mathcal{H}$ satisfies Condition 5, and that the Bohmian guidance equation is T-reversal invariant for the free particle Hamiltonian. Then the action of T on $\mathcal{H}$ must be antilinear, and either $Tx = x$, or else $Tx = x_0 - x$ for some fixed x_0.*

Proof. The assumption that free motion is T-reversal invariant implies that the time reversed values of x and the wavefunction $\psi(x)$ satisfy the Bohmian guidance equation:

$$\frac{dTx}{dt} = -\frac{1}{m}\mathrm{Im}\frac{(T\nabla)\left\langle \hat{T}\mathbf{x}, \hat{T}\psi\right\rangle}{\left\langle \hat{T}\mathbf{x}, \hat{T}\psi\right\rangle}. \tag{14}$$

We will first show how this equation can be simplified considerably. Our conclusion will then follow immediately from the assumption that T is an involution.

Condition 5 requires that the action $\hat{T}$ of T be either linear or antilinear. This means that the wavefunction can only be time reversed in one of two ways: either $\left\langle \hat{T}\varphi_{x_0}, \hat{T}\psi\right\rangle = \langle \varphi_{x_0}, \psi\rangle$ (if the action of time reversal is linear), or $\left\langle \hat{T}\varphi_{x_0}, \hat{T}\psi\right\rangle = \langle \varphi_{x_0}, \psi\rangle^*$ (if the action is antilinear). If $\hat{T}$ is linear, then we can expand ψ in the position basis to get:

$$\begin{aligned}
\left\langle \hat{T}\varphi_{x_0}, \hat{T}\psi\right\rangle &= \left\langle \hat{T}\varphi_{x_0}, \hat{T}\int \langle\varphi_x, \psi\rangle\, \varphi_x\, d\varphi_x \right\rangle && \text{(Expanding } \psi \text{ in the position basis)}\\
&= \left\langle \hat{T}\varphi_{x_0}, \int \langle\varphi_x, \psi\rangle\, \hat{T}\varphi_x\, d\varphi_x \right\rangle && \text{(Linearity of } \hat{T})\\
&= \int \langle\varphi_x, \psi\rangle \left\langle \hat{T}\varphi_{x_0}, \hat{T}\varphi_x\right\rangle d\varphi_x && \text{(Linearity of inner product)}\\
&= \int \langle\varphi_x, \psi\rangle\, \delta(\varphi_x - \varphi_{x_0}) d\varphi_x && \text{(Orthonormality)}\\
&= \langle\varphi_{x_0}, \psi\rangle && \text{(Definition of } \delta)
\end{aligned}$$

where the orthonormality applied in the penultimate line follows from the requirement of Condition 5 that $\hat{T}$ takes position eigenvectors to position eigenvectors, thus preserving their orthonormality. A symmetric argument shows that $\langle T\varphi_{x_0}, T\psi\rangle = \langle\varphi_{x_0}, \psi\rangle^*$ when T is antilinear.

Next, we observe that by the chain rule,

$$T\nabla = \frac{\partial}{\partial Tx} = \frac{\partial x}{\partial Tx}\frac{\partial}{\partial x} = \frac{\partial x}{\partial Tx}\nabla.$$

Combing these two results, we now find that Equation (14) reduces to:

$$\frac{dTx}{dt} = \pm\left(\frac{dx}{dTx}\right)\frac{1}{m}\mathrm{Im}\frac{\nabla\langle\mathbf{x},\psi\rangle}{\langle\mathbf{x},\psi\rangle},$$

where we get a '$+$' if T is antilinear and a '$-$' if T is linear. But the RHS is just that of the usual guidance equation, with an extra factor $\mp(dx/dTx)$. Therefore, we can substitute in

$$\frac{dx}{dt} = -\frac{1}{m}\mathrm{Im}\frac{\nabla\langle\mathbf{x},\psi\rangle}{\langle\mathbf{x},\psi\rangle},$$

to get that:

$$\frac{dTx}{dt} = \mp\left(\frac{dx}{dTx}\right)\frac{dx}{dt}.$$

Multiplying by inverses, this implies,

$$\frac{dTx}{dt}\frac{dt}{dx}\frac{dTx}{dx} = \left(\frac{dTx}{dx}\right)^2 = \mp 1.$$

But T is real-valued, so the -1 case is impossible. This establishes that T must be an antilinear Hilbert space operator.

Finally, since $dTx/dx = \pm 1$, we find by integration that

$$Tx = x_0 \pm x$$

for some constant x_0. But since we have assumed that T is involution, this reduces to only two options: either $Tx = x$, or else $Tx = x_0 - x$. □

Proposition 5. *Suppose that T is a linear bijection on phase space, that T is an involution ($T^2 = I$), and that the Hamiltonian equations are T-reversal invariant for the free particle Hamiltonian. Then either $Tp = -p$ and $Tq = q$, or else $Tp = p$ and $Tq = q_0 - q$ for some fixed q_0.*

Proof. Substitute $t \mapsto -t$ into Hamilton's equations for a single free particle:

$$-\frac{d}{dt}q(-t) = \frac{1}{m}p(-t) \tag{15}$$

$$\frac{d}{dt}p(-t) = 0. \tag{16}$$

Free Motion Symmetry implies that since $q(t)$ and $p(t)$ form a solution to Hamilton's equations, so do $Tq(-t)$ and $Tp(-t)$. Therefore,

$$\frac{d}{dt}Tq(-t) = \frac{1}{m}Tp(-t) \tag{17}$$

$$\frac{d}{dt}Tp(-t) = 0. \tag{18}$$

We now note that (16) and (18) can be integrated to get $Tp(-t) = cp(-t)$. This tells us how T operates on p. To determine how T operates on q, we substitute $Tp(-t) = cp(-t)$ into (17):

$$\begin{aligned} \frac{d}{dt}Tq(-t) &= \frac{1}{m}cp(-t) \\ &= -c\frac{d}{dt}q(-t) \end{aligned}$$

where we have substituted (15) in the last line. Integrating, we see that $Tq(-t) = -cq(-t) + q_0$ for some constant q_0. Therefore, $Tp = cp$ and $Tq = q_0 - cq$. To complete the proof, we now assume that T is a linear involution. Then:

$$p = T^2p = T(cp) = c^2p,$$

so $c^2 = 1$. Moreover,

$$\begin{aligned} q = T^2q &= T(q_0 - cq) \\ &= q_0(1-c) + q. \end{aligned} \tag{19}$$

The constant term $q_0(1-c)$ must therefore vanish. Since $c^2 = 1$ and c is real, there are two options. If $c = -1$, then $q_0 = 0$. Then $Tp = -p$ and $Tq = q$. On the other hand, if $c = 1$, then any real q_0 will satisfy (19). Then we get a class of time-reversal operators indexed by q_0, namely, $Tp = p$ and $Tq = q_0 - q$. $\square$

References

Albert, D. Z. (2000). *Time and chance*. Harvard University Press.

Allori, V., S. Goldstein, R. Tumulka, and N. Zanghi (2008). On the Common Structure of Bohmian Mechanics and the Ghirardi-Rimini-Weber Theory: Dedicated to GianCarlo Ghirardi on the occasion of his 70th birthday. *British Journal for the Philosophy of Science 59*(3), 353–389.

Arntzenius, F. and H. Greaves (2009). Time reversal in classical electromagnetism. *The British Journal for the Philosophy of Science 60*(3), 557–584.

Ballentine, L. E. (1998). *Quantum Mechanics: A Modern Development*. World Scientific Publishing Company.

Bargmann, V. (1964). Note on Wigner's theorem on symmetry operations. *Journal of Mathematical Physics 5*(7), 862–868.

Bogolubov, N. N., A. A. Logunov, A. I. Oksak, and I. T. Todorov (1990). *General Principles of Quantum Field Theory*, Volume 10 of *Mathematical Physics and Applied Physics*. Dordrecht: Kluwer Academic Publishers. Trans. G. G. Gould.

Bohm, D. (1952). A suggested interpretation of the quantum theory in terms of hidden variables. *Physical Review 85*(2), 166–179.

Callender, C. (2000). Is Time 'Handed' in a Quantum World? In *Proceedings of the Aristotelian Society*, pp. 247–269. Aristotelian Society.

Deotto, E. and G. Ghirardi (1998). Bohmian mechanics revisited. *Foundations of Physics 28*(1), 1–30.

Dürr, D., S. Goldstein, and N. Zanghí (1992). Quantum equilibrium and the origin of absolute uncertainty. *Journal of Statistical Physics 67*(5-6), 843–907.

Earman, J. (2002). What time reversal is and why it matters. *International Studies in the Philosophy of Science 16*(3), 245–264.

Earman, J. (2008). Superselection Rules for Philosophers. *Erkenntnis 69*(3), 377–414.

Ghirardi, G. C., A. Rimini, and T. Weber (1986, Jul). Unified dynamics for microscopic and macroscopic systems. *Physical Review D 34*(2), 470–491.

Kadison, R. V. and J. R. Ringrose (1983). *Fundamentals of the Theory of Operator Algebras*, Volume I: Elementary Theory. New York: Academic Press.

Khriplovich, I. B. and S. K. Lamoreaux (1997). *CP Violation Without Strangeness: Electric Dipole Moments of Particles, Atoms, and Molecules*. Springer-Verlag Berlin Heidelberg.

Kim, Y. S. and M. E. Noz (1986). *Theory and Applications of the Poincaré Group*. Dordrecht: D. Reidel Publishing Company.

Le Bellac, M. (2006). *Quantum Physics*. New York: Cambridge University Press.

Malament, D. B. (2004). On the time reversal invariance of classical electromagnetic theory. *Studies in History and Philosophy of Modern Physics 35*, 295–315.

Melsheimer, O. (1974). Rigged Hilbert space formalism as an extended mathematical formalism for quantum systems. i. general theory. *Journal of Mathematical Physics 15*(7), 902–916.

Merzbacher, E. (1998). *Quantum Mechanics* (3rd ed.). Hamilton Printing Company.

Messiah, A. (1999). *Quantum Mechanics, Two Volumes Bound as One*. Dover.

Pearle, P. (1989, Mar). Combining stochastic dynamical state-vector reduction with spontaneous localization. *Physical Review A 39*(5), 2277–2289.

Sachs, R. G. (1987). *The Physics of Time Reversal*. Chicago: University of Chicago Press.

Sakurai, J. J. (1994). *Modern Quantum Mechanics* (Revised Edition ed.). Addison Wesley.

Shankar, R. (1980). *Principles of Quantum Mechanics* (2nd ed.). New York: Springer Science and Business Media.

Stone, A. D. (1994). Does the Bohm Theory Solve the Measurement Problem? *Philosophy of Science 61*(2), 250–266.

Streater, R. F. (2007). *Lost Causes in and beyond Physics*. Springer-Verlag Berlin Heidelberg.

Tannor, D. J. (2007). *Introduction to quantum mechanics: a time-dependent perspective*. University Science Books.

Wigner, E. (1931). *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra*. New York: Academic Press (1959).

Woodhouse, N. M. J. (1991). *Geometric Quantization*. New York: Oxford University Press.

Zeidler, E. (2009). *Quantum Field Theory*, Volume II: Quantum Electrodynamics. Springer-Verlag Berlin Heidelberg.