The simple failure of Curie's Principle

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Abstract. I point out a simple sense in which the standard formulation of Curies Principle is false, when the symmetry transformation it describes is time reversal.

1. Introduction

John Earman has suggested that there is a simple formulation of Curie's Principle that is not only deeply intuitive, but "virtually analytic" (Earman 2004, p.173). He is not the only one to take this view[1], but gives one of its clearest statements. Earman formulates Curie's Principle as the claim: *If*,

(CP1) the laws of motion/field equations governing the system are deterministic;
(CP2) the laws of motion/field equations governing the system are invariant under a symmetry transformation; and
(CP3) the initial state of the system is invariant under said symmetry; then
(CP4) the final state of the system is also invariant under said symmetry. (Earman 2004, p.176)

Speaking intuitively, one might just say: *if no asymmetry goes in, then no asymmetry comes out.*

[1]For example, Mittelstaedt and Weingartner (2005, p.231) argue, on the tacit assumption that the laws of physics are deterministic, that "from an asymmetric effect and symmetric laws we may conclude asymmetric initial conditions." Ismael (1997, p.170) claims to have *proven* that "all characteristic symmetries of a Curie-cause are also characteristic symmetries of its effect." Curie himself suggests that his principle is an *a priori* truth (Curie 1894); see (Brading and Castellani 2003, p.311-313) for an English translation of his famous article.

I would like to point out a simple sense in which this formulation of Curie's Principle fails, when the symmetry transformation is time reversal. I will begin by illustrating a very simple counterexample in classical Hamiltonian mechanics, and then show how this counterexample is endemic to quantum mechanics and quantum field theory. I conclude by discussing three revised principles, which avoid the counterexample, but do not appear to adequately capture the formulation of Curie's principle expressed above.

2. The simple failure of Curie's Principle

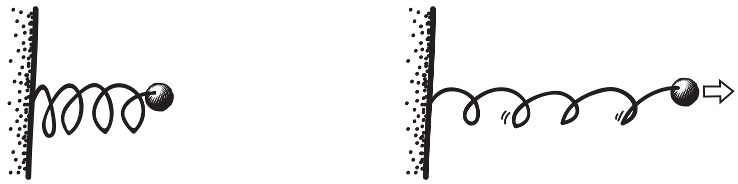
2.1. **In pictures.** Take a harmonic oscillator, such as a bob on a spring. It is manifestly time reversal invariant, in that for every possible motion of the bob, there is a "time-reversed motion" that is also possible.

We must now describe how the instantaneous state of the bob transforms under time reversal. The standard answer, found in any physics book on the topic, specifies that the position of a state remains unchanged, while the direction of the momentum is reversed. The intuition for this just comes from imagining that we film the motion of the bob, and then play the film in reverse. Looking at an instantaneous state and asking what time reversal does is like looking at a single "frame" of the film, and asking how it differs in the past-directed version as opposed to the future-directed version. The answer is that, since rightward motion in the original film becomes leftward in the reversed film, the momenta simply reverse sign. It follows that a state of this system is "invariant" (or "unchanged" or "preserved") under time reversal if and only if the momentum of that state is zero.

Let us now suppose that this particular bob-spring system begins its motion at time $t = 0$ with the spring compressed out of equilibrium, and with no initial momentum, as in Figure 1(a). The bob then springs back in the other direction, acquiring some non-zero momentum, as in Figure 1(b). How does time reversal transform these initial and final states?

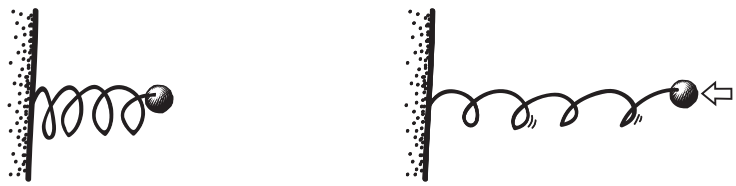
Our initial state has zero momentum, so it is preserved by the time reversal operator. But the final state has non-zero momentum, which reverses direction under the time reversal operator. The result: the laws of motion for the harmonic oscillator are time reversal invariant, and the initial state is preserved by the time reversal operator, but the final state is not. The harmonic oscillator is a system for which Curie's Principle, on this formulation[2], fails.

[2] Alternative formulations of Curie's Principle will be considered in Section 4.



(a) Initial state (b) Final state

Figure 1. *(a) A harmonic oscillator initially compressed out of equilibrium with zero momentum. (b) A final state for which the system has non-zero momentum.*



(a) Time reversed initial state (b) Time reversed final state

Figure 2. *(a) The initial state has no momentum, and so is preserved by time reversal. (b) A final state has non-zero momentum, and so is not preserved by time reversal.*

2.2. **Mathematical verification.** Let's do the exercise of checking this result in Hamiltonian mechanics. The possible states of the harmonic oscillator are the possible values for the position and momentum (q, p) of the bob in phase space. The laws of motion for the system are Hamilton's equations,

$$\frac{d}{dt}q(t) = \frac{\partial}{\partial p}h(q,p), \qquad \frac{d}{dt}p(t) = -\frac{\partial}{\partial q}h(q,p).$$

The Hamiltonian $h(q, p)$ for the harmonic oscillator is $h(q, p) = q^2 + p^2$. The laws of motion are thus manifestly time reversal invariant, in that if $(q(t), p(t))$ is a possible trajectory, then $(q(-t), -p(-t))$ is a possible trajectory as well[3].

[3] To verify: Let $(q(t), p(t))$ be a solution to Hamilton's equations. The Hamiltonian $h(q, p) = p^2 + q^2$ has the property that $h(q, p) = h(q, -p)$. So, Hamilton's equations also hold for $h(q, -p)$. But Hamilton's equations hold for all values of t, and therefore under the substitution $t \mapsto -t$. Making this substitution, we thus find that $-(d/dt)q(-t) = \partial h(q, p)/\partial p$ and hence that $(d/dt)q(-t) = \partial h(q, -p)/\partial(-p)$; similarly, $-(d/dt)p(-t) = -\partial h(q, p)/\partial q$, and hence $(d/dt)(-p(-t)) = -\partial h(q, -p)/\partial q$. That is, $(q(-t), -p(-t))$ is also a solution to Hamilton's equations.

We now need to check that there is a trajectory with an initial state that is preserved by time reversal, and a final state that is not. One such trajectory is the following, which one can check[4] is a solution to the laws of motion above:

$$q(t) = \cos(2t), \quad p(t) = -\sin(2t).$$

At time $t = 0$, this system has zero momentum, since $p(0) = \sin(0) = 0$. But it has non-zero momentum for the subsequent times $0 < t < 2\pi$. The time reversal operator $T : (q, p) \mapsto (q, -p)$ therefore preserves the initial state, but not all later states.

2.3. **Summary.** Here is what we have observed in the example above:

(1) *The harmonic oscillator is time reversal invariant.* This is a simple mathematical fact about the law of motion for the harmonic oscillator.
(2) *The harmonic oscillator has a trajectory for which the initial state is preserved under time reversal.* We choose a trajectory for which the harmonic oscillator is not always in equilibrium, and then choose an initial state with zero momentum.
(3) *Not all later states of the same trajectory are so preserved.* The later states of the harmonic oscillator have non-zero momentum, and so are not preserved by the time reversal operator.

Curie's Principle thus fails when the symmetry transformation is time reversal.

3. Robust failure in quantum theory

Our example above made use of the way that the classical position and momentum variables (q, p) transform under time reversal. But Curie's Principle fails just as badly in quantum theory, and we need not make any mention of position or momentum to show this[5]. I'll begin by describing the standard definition of time reversal and time reversal invariance in quantum theory, and then show how Curie's Principle fails.

[4]Namely, $dq/dt = (d/dt)(\cos(2t)) = -2\sin(2t) = 2p(t) = \partial h/\partial p$, and $dp/dt = (d/dt)(-\sin(2t)) = -2\cos(2t) = -2q(t) = -\partial h/\partial q$.

[5]In fact, there is a similarly robust way to describe this failure in the geometric formulation of classical mechanics (a classic textbook in this formulation is Abraham and Marsden 1978). But that discussion lies outside the scope of the simple point I would like to make here.

3.1. **Time reversal in quantum theory.** Curie's Principle fails quite generally in both non-relativistic quantum mechanics and in relativistic quantum field theory. To keep the discussion general enough to apply to both, I will characterize the spacetime on which quantum theory takes place as an affine space $\mathcal{M}$, which admits a foliation into spacelike hypersurfaces. This will allow us to think of $\mathcal{M}$ as either a non-relativistic spacetime (such as Newtonian or Galilei spacetime), or a relativistic spacetime (such as Minkowski spacetime).

The vector states of a quantum system will be described by vectors in a Hilbert space $\mathcal{H}$. For any foliation Σ_t of the spacetime $\mathcal{M}$ into spacelike hypersurfaces, we take there to be a continuous one-parameter group of unitary opeartors $\mathcal{U}_t = e^{-itH}$. This group describes the way any initial state $\psi \in \mathcal{H}$ changes over time, by the rule,

$$\psi(t) = e^{-itH}\psi.$$

In differential form, this law becomes the familiar Schrödinger equation $i(d/dt)\psi(t) = H\psi(t)$, which holds for all $\psi(t)$ in the domain of H.

Time reversal in quantum mechanics is a transformation that takes a trajectory $\psi(t)$ to a new trajectory $T\psi(-t)$, where $T : \mathcal{H} \to \mathcal{H}$ is a bijection called the *time reversal operator*. This operator T has the special property of being *antiunitary*. An antiunitary operator satisfies $T^*T = TT^* = I$, but it is antilinear instead of linear, meaning that for any two vectors ψ and ϕ and for any complex constants a and b,

$$T(a\psi + b\phi) = a^*T\psi + b^*T\phi. \tag{1}$$

Although unusual, antiunitarity is absolutely essential to capturing the meaning of time reversal in quantum theory; Wigner (1931, §20) remains one of the best discussions of this principle.

A quantum system $(\mathcal{H}, e^{-itH})$ is *time reversal invariant* if, whenever $\psi(t)$ is a solution to the law of motion, then so is $T\psi(-t)$. This is equivalent[6] to the statement,

$$THT^{-1} = H, \tag{2}$$

where H is the generator (the "Hamiltonian") appearing in the unitary dynamics $\mathcal{U}_t = e^{-itH}$.

3.2. **Curie's Principle in quantum theory.** Here is how Curie's Principle goes awry in this theory. Let $(\mathcal{H}, e^{-itH})$ be any time reversal invariant quantum system, in that $THT^{-1} = H$. Suppose the initial state ψ is preserved by the time reversal operator, $T\psi = \psi$. Then it is not generally true that $T\psi(t) = \psi(t)$ for all t.

[6]This was pointed out, for example, in (Earman 2002, p.248).

To see why, notice first that since $T\psi = \psi$, we may bring the T over and write $\psi = T^{-1}\psi$, and thus that $Te^{-itH}\psi = Te^{-itH}T^{-1}\psi$. This allows a simple calculation:

$$T\psi(t) = Te^{-itH}T^{-1}\psi = e^{T(-itH)T^{-1}}\psi = e^{itTHT^{-1}}\psi = e^{itH}\psi = \psi(-t).$$

The second equality follows from the functional calculus[7], the third from the antilinearity of T expressed in Equation (1), and the fourth from the assumption of time reversal invariance expressed in Equation (2).

From this it is clear that later states will be preserved by time reversal (that is, $T\psi(t) = \psi(t)$) if and only if $\psi(t) = \psi(-t)$. In other words, the trajectory $\psi(t)$ would have to be symmetric about initial time $t = 0$. This is not generally the case. Even worse: since Curie's Principle is supposed to hold of any initial state, its satisfaction would imply that $\psi(t) = \psi(-t)$ *for all* states, at all times t. This is only possible if the state $\psi(t) = \psi$ is fixed for all of time[8]. So, Curie's Principle fails for every quantum system that is interesting enough to allow any change whatsoever in time.

In summary: the time reversal invariance of a quantum system $(\mathcal{H}, \mathcal{U}_t)$ implies that $T\mathcal{U}_t\psi = \mathcal{U}_{-t}T\psi$. So, if $T\psi = \psi$, then $T\mathcal{U}_t\psi = \mathcal{U}_{-t}\psi$. This contradicts the conclusion of Curie's Principle, that $T\mathcal{U}_t\psi = \mathcal{U}_t\psi$, in all but the simplest of cases.

4. Revising Curie's Principle

There are at least three ways to revise Curie's Principle to get a true proposition. None seem to me to provide a satisfactory way to capture the principle. Let me discuss each of them in turn.

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[7] There is an easy way to see this without the functional calculus, by restricting attention to the so-called "analytic vectors" of H. Such a vector ψ allows the expansion of the exponential as $e^{-itH}\psi = \sum \frac{(-itH)^k}{k!}\psi$. Since $TT^{-1} = I$, we can write $T(-itH)^kT^{-1} = (-TitHT^{-1})^k$. So, applying T to our expansion we see that $Te^{-itH}T^{-1}\psi = \sum \frac{T(-itH)^kT^{-1}}{k!}\psi = \sum \frac{(-TitHT^{-1})^k}{k!}\psi = e^{T(-itH)T^{-1}}\psi$.

[8] Proof: we will show that for any initial state ψ and for all $t \in \mathbb{R}$, $\psi(t) = \psi$. Let $\psi \in \mathcal{H}$ and let $t \in \mathbb{R}$. Define a new initial state $\phi := e^{-i(t/2)H}\psi$, with $\phi(t) := e^{-itH}\phi$. Curie's principle implies that $\psi(t) = \psi(-t)$ for all trajectories and for all times; hence in particular $\phi(t/2) = \phi(-t/2)$. But $\phi(t/2) = e^{-i(t/2)H}\phi = e^{-i(t/2)H}e^{-i(t/2)H}\psi = \psi(t)$, while $\phi(-t/2) = e^{itH}e^{-itH}\psi = \psi$. Therefore, $\psi(t) = \psi$, which proves the claim.

4.1. **Argue time reversal is not a symmetry.** One way to revise Curie's Principle is to restrict what counts as a "symmetry transformation." By excluding problematic transformations like time reversal, one can produce mathematically correct replacements for Curie's Principle.

Earman himself has formulated one such statement, which he takes to capture Curie's Principle in the algebraic framework for quantum field theory. He begins with a C^* algebra, with an automorphism group α describing the dynamics. His approach is then to characterize a "symmetry transformation" in quantum field theory as (linear) automorphism θ of the C^* algebra. In this framework, Earman writes:

> *Proposition 2 (Curie's Principle).* Suppose that the initial state ω_o is θ-symmetric (i.e. $\widehat{\theta\omega_o} := \omega_o \circ \theta = \omega_o$) and that the dynamics α is also θ-symmetric (i.e. $\theta\alpha\theta^{-1} = \alpha$). Then the evolved state $\omega_1 := \widehat{\alpha\omega_o}$ is θ-symmetric. (Earman 2004, p.198)

This certainly resembles Curie's principle: the dynamics are deterministic (CP1), the dynamics are preserved by a symmetry (CP2), the initial state is preserved by the symmetry (CP3), and we conclude that the final state is preserved by the symmetry (CP4). There is also an easy analogue in non-relativistic quantum mechanics. There, the approach would be to characterize a symmetry transformation θ as a (linear) unitary transformation on a Hilbert space $\mathcal{H}$. Then we have:

> *Non-Relativistic Proposition 2.* Suppose that the initial state $\psi_0 \in \mathcal{H}$ is θ-symmetric (i.e. $\theta\psi_0 = \psi_0$) and that the unitary group e^{-itH} generating the dynamics is also θ-symmetric (i.e. $\theta e^{-itH}\theta^{-1} = e^{-itH}$). Then the evolved state $\psi_1 := e^{-itH}\psi_0$ is θ-symmetric.

Both of these propositions are mathematically correct, and their proofs are trivial[9]. Time reversal is excluded from the content of both propositions, because the time reversal operator in quantum theory is not linear but *anti*linear; see Section 3.1.

Although Earman's approach saves a Curie-like principle, it is at the expense of the orthodox definition of symmetry transformations in quantum theory. In quantum theory, symmetry transformations include not only the linear-unitary transformations, but the *anti*linear-*anti*unitary transformations as well. In the algebraic framework in

[9]Earman states the former; the latter is similar: $\theta\psi_1 = \theta e^{-itH}\psi_0 = e^{-itH}\theta\psi_0 = e^{-itH}\psi_0 = \psi_1$

which Earman works, symmetry transformations include both linear-automorphisms and antilinear-anti-automorphisms. This is the orthodox view of symmetries, arising out of Wigner's theorem and its generalizations[10]. So, these revised statements fall short of capturing the original statement of Curie's Principle, in excluding an important class of orthodox symmetries.

A clever next response is to notice that, although Earman's discussion does not mention antilinear operators, the above two propositions actually do hold when θ is antilinear! (Their proofs go through in the very same way.) However, when $\theta = T$ is the time reversal operator, the premise that $\theta e^{-itH}\theta^{-1} = e^{-itH}$ (or $\theta\alpha\theta^{-1} = \alpha$ in the relativistic version) does *not* capture the usual notion of "invariance" under time reversal in quantum theory. As we saw in Section 3.1, time reversal invariance is equivalent to the statement that $THT^{-1} = H$. But since T is antilinear, this implies that

$$Te^{-itH}T^{-1} = e^{T(-itH)T^{-1}} = e^{itTHT^{-1}} = e^{itH}.$$

That is, time reversal invariance does not mean that the dynamics are unchanged, but that the temporal order is reversed. This observation suggests that one might rescue Curie's principle by modifying what it means for the laws of motion to be "invariant" under a transformation. This is the option that I will describe next.

4.2. **Argue for a non-standard notion of invariance.** This response argues for a non-standard notion of "invariance" of the laws of motion under a symmetry transformation. This strategy is to modify Earman's statement (CP2) along the following lines.

(CP2′) If an initial state is invariant under a symmetry transformation, then so is the final state.

This statement, together with Earman's statement (CP3) "the initial state of the system is invariant under said symmetry transformation," obviously implies (CP4): "the final state of the system is also invariant under said symmetry transformation." Thus we have another correct statement that resembles Curie's Principle. And we have already seen a mathematical expression of it in quantum theory: it is simply the above Proposition 2 (in relativistic or non-relativistic form) opened up to allow both linear and antilinear transformations.

[10]Cf. (Wigner 1931, §20), (Uhlhorn 1963), (Varadarajan 2007, Theorem 4.29); the latter two take a symmetry to be an automorphism of the lattice of projections, which extend to both automorphisms and the anti-automorphisms of the C^* algebra.

Unfortunately, this does not appear to me to be an interesting way to capture Curie's Principle, because it is not the standard meaning of "invariance" or "symmetry" of the laws, and because it renders Curie's Principle so trivial it runs the risk of being uninteresting.

First, on the "invariance" or "symmetry" of a law under a transformation: when cashed out precisely[11], this is always taken to mean that the set of models (or trajectories) of the laws are preserved. In other words, a deterministic law is invariant under a transformation if, whenever a trajectory $\psi(t)$ is a solution to the law, so is the transformed trajectory $\psi'(t')$. This is the meaning that Earman has in mind in his statement (CP2), stating that a symmetry "carries solutions to solutions" (Earman 2004, p.176). If we follow Earman in interpreting the notion of invariance/symmetry of the laws in the standard way, then we do not get (CP2′).

More importantly, this approach to Curie's Principle is so trivial as to appear devoid of content. The expression, "$(CP2')$ and $(CP3)$ implies $(CP4)$" is just a tautology of the form, "$A \to B$ and A implies B." This is more of a linguistic inference rule than it is a symmetry principle relating the symmetries of states and laws, as Curie's Principle is generally taken to be. Some may be content to trivialize the statement in this way. But I do not think this is what Curie himself had in mind. Curie wrote:

> When certain causes produce certain effects, the symmetry elements of the causes must be found in the produced effects (Curie 1894, p.401)[12].

If by "causes" Curie means the laws and the initial state, and if by "symmetry" he means the standard interpretation discussed above, then Curie's Principle is best stated along the lines that Earman has formulated in (CP1)-(CP4), and not by the revision expressed in (CP2′).

4.3. **Argue that Curie's Principle is about trajectories.** A third response is to retain the orthodox definitions of symmetry and invariance, but to modify the kind of object that Curie's Principle is about. The last premise and the conclusion of Curie's Principle (Earman's CP3 and CP4) are about *states*. They read:

(CP3) the initial state of the system is invariant under the symmetry;
(CP4) the final state of the system is invariant under said symmetry.

[11]When precision is sacrificed, this phrase is sometimes taken to mean that the "form" of the laws does not change under a transformation.

[12]English translation from (Brading and Castellani 2003, p.312).

But premise (CP2) is about invariance of the laws, which on the standard interpretation refers to *an entire trajectory*. In particular (as discussed in the previous subsection), the laws are invariant under a transformation if whenever $\psi(t)$ is a possible trajectory, then so is the transformed trajectory $\psi'(t')$. So, we can view the trouble with Curie's principle as one of discord between two objects interest: states in one premise, and trajectories in another.

One can bring these objects of interest into closer agreement by making *all* the premises of Curie's principle about trajectories. To do this, let us write $\{\psi(t) = e^{-itH}\psi \mid t \in \mathbb{R}\}$ to denote the trajectory with initial state ψ. We begin by distinguishing two senses in which a state $\psi(t)$ in that trajectory can be "symmetric" with respect to a symmetry transformation.

(1) A state $\psi(t)$ at a time t is *S-symmetric in the original order* if $S\psi(t) = \psi(t)$.
(2) A state $\psi(t)$ at a time t is *S-symmetric in the reverse order* if $S\psi(t) = \psi(-t)$.

This is not such an unusual distinction, when one recalls (from the end of Section 4.1) that the standard definition of time reversal invariance entails a similar reversal of sign: $Te^{-itH}T^{-1} = e^{itH}$.

We can now express a revision of Curie's Principle: *If*,

(CP1) the laws of motion/field equations governing the system are deterministic;
(CP2) the laws of motion/field equations governing the system are invariant under a symmetry transformation; and
(CP3′) the state of the system at some fixed time t_0 is symmetric under said symmetry (in the original or reverse order);
then,
(CP4′) the state of the system at any time t is symmetric under said symmetry (in the same order).

In the context of ordinary quantum mechanics, this statement corresponds to the following two facts[13].

Fact 1. Suppose a state $\psi(t_0) := e^{-it_0H}\psi$ at a fixed time t_0 is θ-symmetric in the original order (i.e. $\theta\psi(t_0) = \psi(t_0)$), and that the unitary group e^{-itH} generating the dynamics is invariant under θ in the original order (i.e. $\theta e^{-itH}\theta^{-1} = e^{-itH}$). Then for all times t, the state $\psi(t) = e^{-itH}\psi$ is θ-symmetric in the same order.

[13]Fact 1 follows from the non-relativistic version of Proposition 2 in the last subsection. Fact 2 is proved: $T\psi(t) = Te^{-i(t-t_0)H}e^{-it_0H}\psi = Te^{-i(t-t_0)H}\psi(t_0) = e^{i(t-t_0)H}T\psi(t_0) = e^{i(t-t_0)}\psi(-t_0) = e^{i(t-t_0)H}e^{it_0H}\psi = e^{itH}\psi = \psi(-t)$.

Fact 2. Suppose a state $\psi(t_0) := e^{-it_0 H}\psi$ at a fixed time t_0 is θ-symmetric in the reverse order (i.e. $\theta\psi(t_0) = \psi(-t_0)$), and that the unitary group e^{-itH} generating the dynamics is invariant under θ in the reverse order (i.e. $\theta e^{-itH}\theta^{-1} = e^{itH}$). Then for all times t, the state $\psi(t) = e^{-itH}\psi$ is θ-symmetric in the reverse order.

We have again arrived at a correct mathematical statement. Time reversal is no longer excluded, being captured now by Fact 2. We have moreover retained the usual definition of a "symmetry/invariance" of the laws. But is this Curie's Principle? Strictly speaking, Curie's Principle says that if the initial state is *preserved* by a symmetry transformation, then so is the final state. This is not what is described by Fact 2 above, where the symmetry transformation "flips" each state about the temporal origin. Facts 1 and 2 perhaps express a more natural principle, in bringing the premises into closer alignment. But they do not capture the original expression of Curie's Principle.

5. Conclusion

There does not appear to be much hope for the standard statement of Curie's Principle formulated in the first section. Time reversal provides a simple counterexample. We have seen that there remain statements *like* Curie's Principle that are mathematically correct. They can be achieved either by excluding symmetry transformations like time reversal, or by modifying the statements (CP2)-(CP4) appearing in the principle. However, the standard statement of Curie's Principle, given the standard meaning of the language therein, is false. It is false in classical Hamiltonian mechanics, false in quantum mechanics. This appears to be a dramatic failure indeed.

References

Abraham, R. and Marsden, J. E. (1978). *Foundations of Mechanics*, 2nd edn, Addison-Wesley Publishing Company, Inc.

Brading, K. and Castellani, E. (2003). *Symmetries in physics: philosophical reflections*, Cambridge: Cambridge University Press.

Curie, P. (1894). Sur la symétrie dans les phénomènes physique, symétrie d'un champ électrique et d'un champ magnétique, *Journal de Physique Théorique et Appliquée* **3**: 393–415.

Earman, J. (2002). What time reversal is and why it matters, *International Studies in the Philosophy of Science* **16**(3): 245–264.

Earman, J. (2004). Curie's Principle and spontaneous symmetry breaking, *International Studies in the Philosophy of Science* **18**(2-3): 173–198.

Ismael, J. (1997). Curie's Principle, *Synthese* **110**(2): 167–190.
Mittelstaedt, P. and Weingartner, P. A. (2005). *Laws of Nature*, Springer-Verlag Berlin Heidelberg.
Uhlhorn, U. (1963). Representation of symmetry transformations in quantum mechanics, *Arkiv för Fysik* **23**: 307–340.
Varadarajan, V. S. (2007). *Geometry of Quantum Theory*, New York: Springer Science and Business Media, LLC.
Wigner, E. P. (1931). *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra*, New York: Academic Press (1959).