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# The Scope and Generality of Bell's Theorem

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**Abstract**

I present a local, deterministic model of the EPR-Bohm experiment, inspired by recent work by Joy Christian, that appears at first blush to be in tension with Bell-type theorems. I argue that the model ultimately fails to do what a hidden variable theory needs to do, but that it is interesting nonetheless because the way it fails helps clarify the scope and generality of Bell-type theorems. I formulate and prove a minor proposition that makes explicit how Bell-type theorems rule out models of the sort I describe here.

## 1 Introduction

Bell-type theorems are usually understood to rule out a class of hidden variable interpretations of quantum mechanics that are, in a certain precise sense, local and deterministic. These theorems come in a variety of forms, with different characterizations of what local and deterministic are meant to amount to. In the simplest case, Bell-type theorems show that any hidden variable model satisfying some set of conditions fails to agree with the quantum mechanical predictions for the outcomes of a simple experiment, often called the EPR-Bohm experiment. When one performs the suggested experiments, meanwhile, the results agree with quantum mechanics. The upshot appears to be that no local, deterministic hidden variable model can correctly reproduce the measurement outcomes of a certain class of experiments.[2]

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[2] For an excellent over view of the state of the art on Bell-type theorems, see Shimony (2009). See also Jarrett (1984, 1989) and Malament (2006) for particularly clear expositions of how Bell-type theorems work, with the relevant assumptions stated as precisely as possible. I should note, though, that the version of

In this paper, I discuss a geometrical model of EPR-Bohm experiments that appears, at least at first blush, to side-step Bell-type theorems.[3] The model offers an explicit representation of the states of the particles in an EPR-Bohm experiment. It is manifestly local and fully deterministic. And one can argue that it yields the correct measurement outcomes, as predicted by quantum mechanics. As one would expect given what I have just said, the model I will describe differs from the kind of system Bell seemed to have in mind in several important ways—ways, I might add, that are physically motivated by the EPR-Bohm experiment itself.

Unfortunately, the model does not work. But it seems to me that it fails in an interesting way. The model draws attention to just what criteria a local, deterministic hidden variable model would need to satisfy in order to accurately reproduce the results of an EPR-Bohm experiment. The particular way in which it fails to meet those criteria speaks to the generality of Bell-type theorems. Taking the model as a guide, I will formulate and prove a minor proposition to the effect that a class of hidden variable models similar in a certain respect to the one I describe cannot in fact reproduce the EPR-Bohm measurement results, properly understood. Remarkably, the proposition follows as a corollary of a version of Bell's theorem. This means that models of the type I will describe are already ruled out by Bell-type considerations, despite the apparent differences between the model I will discuss and the kind of hidden variable theories with which Bell-type theorems seem to be concerned.[4]

The paper will proceed as follows. I will begin by describing the EPR-Bohm setup and stating the version of Bell's theorem that I will focus on here. Then I will present the model that will be the focus of the present discussion. I will show that it is local and deterministic,

Bell's theorem I will present in section 2 of the present paper is due to Clauser et al. (1969) and is somewhat different than the versions discussed by Jarrett and Malament (which are in the tradition of Clauser and Horne (1974)).

[3]The model I discuss here is inspired by, but not the same as, models discussed by Joy Christian (2007a,c,b, 2008, 2009, 2010, 2011b,a). To my mind, at least, the present model is considerably simpler than Christian's, and it avoids certain technical complications that his models encounter. That said, although my engagement with Christian's work will be from an oblique angle, what I say applies equally well to the models he discusses.

[4]The character of the response offered here is related to a brief criticism of Christian's proposal by Grangier (2007), although the details are quite different.

but that it nonetheless seems to reproduce the quantum mechanical predictions for the EPR-Bohm experiment. The goal here will be to make the case for the model as strongly as possible, since I want to emphasize that there really is something to the idea even though it is ultimately unsuccessful. In the penultimate section, I will describe what goes wrong with the model and then formulate and prove the no-go result described above. I will conclude with some remarks concerning the generality of Bell-type theorems.

## 2 The EPR-Bohm experiment and a Bell-type Theorem

The version of Bell's theorem I will discuss here is based on an experimental configuration known as the EPR-Bohm setup, which involves two spin 1/2 particles, $A$ and $B$, traveling in opposite directions. One assumes that the pair initially has vanishing total spin, which means that the quantum mechanical spin state for the system, $|\Psi\rangle$, is an entangled superposition often called the "singlet state." A natural way to express the singlet state is in terms of the eigenvectors of the spin operator about an arbitrary vector $\mathbf{n}$. The spin operator about $\mathbf{n}$ is given by $\boldsymbol{\sigma}\cdot\mathbf{n}$, where $\boldsymbol{\sigma}$ is the Pauli spin "vector," defined by $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, with $\sigma_x, \sigma_y, \sigma_z$ the Pauli spin matrices.[5] These eigenvectors can be written in the standard Dirac notation so that $\boldsymbol{\sigma}\cdot\mathbf{n}|\mathbf{n},\pm\rangle = \pm|\mathbf{n},\pm\rangle$. The combined spin state of the two-particle system can then be expressed by

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\mathbf{n},+\rangle_A \otimes |\mathbf{n},-\rangle_B - |\mathbf{n},-\rangle_A \otimes |\mathbf{n},+\rangle_B\right),$$

where the $A$ and $B$ subscripts indicate membership in the Hilbert spaces associated with particles $A$ and $B$ respectively.

The experiment consists of two measurements, one on each particle, performed at remote locations by two observers, Alice and Bob. Alice and Bob each choose a vector, $\mathbf{a}$ and $\mathbf{b}$ respectively, and then perform measurements of the spin of their assigned particle about

[5] The Pauli spin vector is actually a map from vectors in three dimensional Euclidean space to the space of operators on the two-dimensional Hilbert space representing spin states. But the abuse of notation should be harmless, since the inner product is shorthand for the obvious thing: $\boldsymbol{\sigma}\cdot\mathbf{n} = \sigma_x n_x + \sigma_y n_y + \sigma_z n_z$. Note that we are working in units where $\hbar = 2$.

the chosen vector (suppose Alice is measuring particle $A$ and Bob is measuring particle $B$). Simple calculations yield the following quantum mechanical predictions for the expected outcomes of these experiments. The expectation values for Alice and Bob's individual experimental outcomes are,

$$\mathcal{E}_{q.m.}(A(\mathbf{a})) = \langle\Psi|(\boldsymbol{\sigma}\cdot\mathbf{a})_A\otimes\mathbb{I}_B|\Psi\rangle = 0 \tag{2.1}$$

$$\mathcal{E}_{q.m.}(B(\mathbf{b})) = \langle\Psi|\mathbb{I}_A\otimes(\boldsymbol{\sigma}\cdot\mathbf{b})_B|\Psi\rangle = 0. \tag{2.2}$$

Meanwhile, the expectation value of the joint observable $(\boldsymbol{\sigma}\cdot\mathbf{a})_A\otimes(\boldsymbol{\sigma}\cdot\mathbf{b})_B$, a measure of the correlation between the two measurements, is given by

$$\mathcal{E}_{q.m.}(A(\mathbf{a}), B(\mathbf{b})) = \langle\Psi|(\boldsymbol{\sigma}\cdot\mathbf{a})_A\otimes(\boldsymbol{\sigma}\cdot\mathbf{b})_B|\Psi\rangle = -\mathbf{a}\cdot\mathbf{b}. \tag{2.3}$$

To interpret this joint expectation value, it is useful to note in particular that if Alice and Bob choose to perform their measurements about orthogonal vectors, their outcomes should not exhibit any correlation at all (they are equally likely to get the same results as opposite results). Conversely, if Alice and Bob choose the same vector, their measurement results should be perfectly anti-correlated (any time Alice yields +1, Bob necessarily will yield -1, and vice versa). These predictions are well-corroborated by experiment.

Bell-type theorems are intended to rule out the possibility that the results of EPR-Bohm experiments can be recovered from a model that somehow represents the "complete" state of the two-particle system in such a way that the results of Alice and Bob's measurements can be understood to be local and deterministic. The strategy is to postulate that there is an unknown (hidden) variable, $\lambda$, such that, were one to specify the value of $\lambda$, the results of Alice and Bob's respective measurements would be (a) wholly determined by $\lambda$ and the choice of his or her measurement vector (call this "determinism"), and (b) independent of the other experimenter's choice of measurement vector (call this "locality"). One then tries to show that no such model could correctly reproduce the predictions of quantum mechanics.

This idea can be made more precise as follows.[6] We postulate that, associated with each of Alice's and Bob's measuring devices are "observables" $A$ and $B$, where by observable we mean simply maps $A, B : \mathbb{R}^3 \times \Lambda \rightarrow \{-1, 1\}$ that take as input (1) the vector about which Alice or Bob is measuring, and (2) an element $\lambda$ of the space $\Lambda$ of complete states of the two particle system. The specification $\lambda$ of the complete state of the two-particle system is what we interpret as the hidden variable. The measurement results, meanwhile, are represented by the integers $\pm 1$, just as in the quantum mechanical case (where the measurement records correspond to the eigenvalues of the relevant observable).

Given such a pair of observables $A$ and $B$, we can define the expectation values for Alice and Bob's measurements by postulating some probability density $\rho : \Lambda \rightarrow [0, 1]$, intended to reflect our ignorance of the actual value of the hidden variable. The expectation values for Alice and Bob's individual measurements will then be given by:[7]

$$\mathcal{E}_{h.v.}(A(\mathbf{a})) = \int_\Lambda A(\mathbf{a}, \lambda)\rho(\lambda)d\lambda$$
$$\mathcal{E}_{h.v.}(B(\mathbf{b})) = \int_\Lambda B(\mathbf{b}, \lambda)\rho(\lambda)d\lambda.$$

The version of Bell's theorem to be stated presently will be formulated as a constraint on the joint expectation values of $A$ and $B$,

$$\mathcal{E}_{h.v.}(A(\mathbf{a}), B(\mathbf{b})) = \int_\Lambda A(\mathbf{a}, \lambda)B(\mathbf{b}, \lambda)\rho(\lambda)d\lambda.$$

As in the quantum mechanical case, this joint expectation is understood as a measure of the correlation between Alice and Bob's measurements.

This discussions suggests the following definitions.

[6] Here I am following in the tradition of Bell (1964) and Clauser et al. (1969). In section 5 I will briefly connect the ideas discussed here to more modern, probabilistic presentations of Bell-type theorems in the tradition of Clauser and Horne (1974).

[7] In the following expressions, I am writing integrals over $\Lambda$ without being fully specific about what $\Lambda$ looks like, with the possible consequence that the interpretation of the expressions is ambiguous. It turns out, though, that Bell-type results are based on features of integrals that are so basic that it does not matter what kind of integral is being written.

**Definition 2.1** *A* hidden variable model *of the EPR-Bohm experiment is an ordered quadruple* $(\Lambda, A, B, \rho)$*, where* $\Lambda$ *is the set of complete states of the system,* $A, B : \mathbb{R}^3 \times \Lambda \to \{-1, 1\}$ *are maps from detector settings and complete state specifications to measurement outcomes for Alice and Bob respectively, and* $\rho : \Lambda \to [0, 1]$ *is a probability density function on the space of complete states.*

The constraints above, of determinism and locality, can then be understood as constraints on $A$ and $B$ for a given hidden variable model.

**Definition 2.2** *A hidden variable model* $(\Lambda, A, B, \rho)$ *is* deterministic *if* $A$ *and* $B$ *are well-defined as functions, in the sense that a specification of a vector in Euclidean space and a hidden variable state uniquely and in all cases determines an element of the codomain of the observables.*

**Definition 2.3** *A hidden variable model* $(\Lambda, A, B, \rho)$ *is* local *if* $A$ *does not vary with Bob's choice of* $\mathbf{b}$*, and likewise,* $B$ *does not vary with Alice's choice of* $\mathbf{a}$*.*

Given these definitions, we can now state the Bell-type theorem that we will focus on in the next section. It is essentially the Clauser-Horne-Shimony-Holt theorem.

**Theorem 2.4 (Clauser et al. (1969))** *Let* $(\Lambda, A, B, \rho)$ *be a local, deterministic hidden variable model of the EPR-Bohm experiment. Then for any choices* $\mathbf{a}$ *and* $\mathbf{a}'$ *for Alice's detector setting, and any choices* $\mathbf{b}$ *and* $\mathbf{b}'$ *for Bob's detector setting, the joint expectation values for Alice and Bob's measurements must satisfy the Clauser-Horne-Shimony-Holt inequality,*

$$|\mathcal{E}_{h.v.}(A(\mathbf{a}), B(\mathbf{b})) - \mathcal{E}_{h.v.}(A(\mathbf{a}), B(\mathbf{b}')) + \mathcal{E}_{h.v.}(A(\mathbf{a}'), B(\mathbf{b})) + \mathcal{E}_{h.v.}(A(\mathbf{a}'), B(\mathbf{b}'))| \leq 2. \quad (2.4)$$

It is easy to verify that there are choices for $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$, and $\mathbf{b}'$ such that the quantum mechanical expectation values defined in Eq. (2.3) violate this inequality, i.e., choices such that

$$|\mathcal{E}_{q.m.}(A(\mathbf{a}), B(\mathbf{b})) - \mathcal{E}_{q.m.}(A(\mathbf{a}), B(\mathbf{b}')) + \mathcal{E}_{q.m.}(A(\mathbf{a}'), B(\mathbf{b})) + \mathcal{E}_{q.m.}(A(\mathbf{a}'), B(\mathbf{b}'))| > 2.$$

This result yields an immediate corollary of Thm. 2.4.

**Corollary 2.5** *There does not exist a local, deterministic hidden variable model of the EPR-Bohm experiment that reproduces the quantum mechanical expectation values.*

## 3 A local, deterministic model after all?

As promised above, I will now explicitly exhibit a local, deterministic hidden variable model of the EPR-Bohm experiment. It will consist of a space $\Lambda$ of complete states, along with a pair of local, deterministic observables $A(\mathbf{a}, \lambda)$ and $B(\mathbf{b}, \lambda)$ that, taken together, yield the correct quantum mechanical expectation values. Note that this means they will maximally violate the Clauser-Horne-Shimony-Holt inequality for the same choices of $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$, and $\mathbf{b}'$ as the quantum mechanical observables. As will become clear, this model is not a hidden variable model in the strict sense of definition 2.1. But one might reasonably argue that it has all of the virtues that one would hope a (generalized) hidden variable model would have.

Before I say what the model is, let me motivate it a little. The experiment whose outcomes we are trying to reproduce involves spin measurements. Quantum mechanical spin does not have a direct classical analogue, but the spin operators satisfy the same algebraic relations as rotations in three dimensional Euclidean space. This is suggestive: one might take it to imply that the right way of thinking about spin is as some sort of rotation. Following this idea, one might reason that when an experimenter measures a particle to be spin up or spin down about a given vector, she is actually measuring the orientation of the body's rotation.

Suppose this idea is right and the experimenters are attempting to determine the orientation of each body's rotation about freely chosen vectors $\mathbf{a}$ and $\mathbf{b}$. How should the experimenters represent their measurement results? Clearly, one way of doing so would be to assign the numbers $\pm 1$ to the measurement results, where, say, $+1$ is understood to correspond to spin up, which would be a counterclockwise rotation about the measurement vector (relative to a right-hand rule), and $-1$ would correspond to spin down and clockwise rotation. This is how the quantum mechanical expectation values come out; it is also how possible measurement results are represented in Bell-type theorems. But there is a case to be made that this is an inappropriate choice. Rotations in three dimensional space have more complicated algebraic and topological structure than the set $\{-1, +1\} \subset \mathbb{R}$. When one represents the results of an EPR-Bohm experiment by $\pm 1$, this structure is lost. One might argue that it is no wonder that a hidden variable models of the type Bell describes cannot adequately represent the results of an EPR-Bohm experiment, since Bell effectively assumes a representational scheme that does not have the expressiveness necessary to capture the physics of the experiment.

There are several ways to represent rotations (or perhaps better, instantaneous states of rotation of a body) in three dimensional space that do respect the appropriate algebraic and topological properties of physical rotations. A particularly natural choice is to use antisymmetric rank 2 tensors. To see how these are a candidate for representing rotation, consider the following. Pick a plane and consider two linearly independent covectors in that plane. Call them $\xi_a$ and $\eta_a$.[8] Now note that there are two orientations of rotations within your chosen plane, clockwise and counterclockwise. These can be thought of as corresponding to the two ways of ordering $\xi_a$ and $\eta_a$: one of the two orientations corresponds to a rotation that begins with $\xi_a$ and sweeps in the direction of $\eta_a$, and the other corresponds to a rotation that begins with $\eta_a$ and sweeps in the direction of $\xi_a$. (Think, for instance, of a clock face. Clockwise rotation corresponds to the minute hand starting at 12 and sweeping towards 1; counterclockwise orientation corresponds to the minute hands starting at 1 and sweeping towards 12.) These two ways of ordering $\xi_a$ and $\eta_a$ can be represented using a pair of related antisymmetric rank 2 tensors, $\overset{1}{\omega}_{ab} = \xi_{[a}\eta_{b]} = \xi_a\eta_b - \xi_b\eta_a$ and $\overset{2}{\omega}_{ab} = \eta_{[a}\xi_{b]} = -\overset{1}{\omega}_{ab}$. Indeed, in three dimensions, any antisymmetric rank 2 tensor can be represented as the antisymmetric product of two vectors, and so any antisymmetric rank 2 tensor represents a rotation in just this way, as an ordering of two vectors in a plane. This means that to associate an antisymmetric rank 2 tensor with a body is to represent the body as having both a plane of rotation (or equivalently, an axis of rotation) and an orientation of rotation within that

[8] At this point, I am changing notations, since the model I will presently describe is most naturally expressed using tensor fields defined on a manifold. Here and throughout I will use the abstract index notation explained in Penrose and Rindler (1987) and Malament (2011). In most cases, one can get away with thinking of the indices as the counting indices of coordinate notation. I will use the following translation manual for vectors previously discussed: I will now use $\alpha^a$ instead of $\mathbf{a}$ to represent the vector about which Alice chooses to make her measurement, and $\beta^a$ instead of $\mathbf{b}$ to represent the vector about which Bob chooses to make his measurement. These vectors still live in $\mathbb{R}^3$.

plane.[9]

We are thus led to the following suggestion. If one wants to adequately represent the rotation of a body about a given axis, taking full account of the algebraic and topological properties exhibited by the group of rotations in three dimensional space, one should take the observables associated with the EPR-Bohm measurement outcomes to be antisymmetric rank 2 tensors.[10] This is particularly important, one might add, since we are ultimately interested in the correlation between the two observables. Hence we want to be careful to capture the full structure of what is being observed, since one might expect that the algebraic properties of rotations are important in understanding how the outcomes of two measurements of rotation will relate to one another. Of course, taking such objects as observables moves one away from a hidden variable model as defined in section 2. But that need not be a problem, especially if it turns out that all one needs to do to find a local, deterministic interpretation of quantum mechanics is to expand the mathematical playing field slightly.

Given this background, the model I propose goes as follows. Our starting point will be the Euclidean manifold $(\mathbb{R}^3, g_{ab})$, where $g_{ab}$ is a flat metric. The space of constant vector fields on the manifold is isomorphic to the tangent space at any one point (by parallel transport); I will conflate these two spaces and refer to both simply as $\mathbb{R}^3$. One can likewise define a space of (constant) antisymmetric rank 2 tensors by taking the antisymmetric product of the

[9]What does it mean to say that representing rotations as antisymmetric rank 2 tensors respects the "algebraic and topological properties" of rotations in three space? The space of constant antisymmetric rank 2 tensors on three dimensional Euclidean space (understood as a vector space) is itself a three dimensional vector space over $\mathbb{R}$ (i.e., it is closed under addition and scalar multiplication) and moreover forms a Lie algebra with the Lie bracket defined by, for any two antisymmetric rank 2 tensors $\overset{1}{\omega}_{ab}$ and $\overset{2}{\omega}_{ab}$, $[\overset{1}{\omega}, \overset{2}{\omega}]_{ab} = \overset{1}{\omega}_{an}\overset{2}{\omega}{}^{n}{}_{b} - \overset{2}{\omega}_{an}\overset{1}{\omega}{}^{n}{}_{b}$. A short calculation shows that this Lie algebra is none other than $\mathfrak{so}_3$, the Lie algebra associated with the Lie group of rotations on three dimensional vectors, $SO(3)$. The elements of $SO(3)$ correspond to the length preserving maps between vectors in $\mathbb{R}^3$ that are standardly identified with rotations. Note that since the Lie algebra $\mathfrak{so}_3$ is isomorphic to the Lie algebra $\mathfrak{su}_2$, the Pauli spin matrices, the operators associated with spin for a spin 1/2 system, are a representation of $\mathfrak{so}_3$.

[10]There are other ways one might do this, too. Christian, for instance, proposes that one represent rotations using bivectors in a Clifford algebra. There are various reasons to think that that proposal is equally well-motivated, or even better motivated, than the one I make here. But ultimately such distinctions will not matter. See the next section for more on this point.

elements of $\mathbb{R}^3$, as described above. I will call this space of tensors $\mathbb{R}^3 \wedge \mathbb{R}^3$. Note that there are automatically two volume elements (normalized antisymmetric rank 3 tensor fields) on the manifold. Call these $\epsilon_{abc}$ and $-\epsilon_{abc}$. We will fix a sign convention up front and take $\epsilon_{abc}$ to represent the right handed volume element.[11]

The hidden variable space for the model will be $\Lambda = \{1, -1\}$. We can then define two observables $A, B : \mathbb{R}^3 \times \Lambda \to \mathbb{R}^3 \wedge \mathbb{R}^3$ as follows:

$$A_{bc}(\alpha^a, \lambda) = \frac{\lambda}{\sqrt{2}} \epsilon_{abc} \alpha^a$$
$$B_{bc}(\beta^a, \lambda) = -\frac{\lambda}{\sqrt{2}} \epsilon_{abc} \beta^a$$

Note that these observables are both antisymmetric rank 2 tensors, as we wanted. They can be understood to represent rotations about the vectors $\alpha^a$ and $\beta^a$, as can be seen by noting that $A_{bc}(\alpha^a, \lambda)\epsilon^{abc} \propto \alpha^a$ and $B_{bc}(\beta^a, \lambda)\epsilon^{abc} \propto \beta^a$. Note, too, that for a particular choice of $\lambda$, and for measurements about parallel vectors, we find

$$A_{bc}(\xi^a, \lambda) = -B_{bc}(\xi^a, \lambda).$$

This means the two measurement results are always anti-correlated, as expected from the quantum mechanical case. Finally, $A$ and $B$ are explicitly local and deterministic: given a value for the hidden variable, and given a choice of measurement vector, the observables take unique, determinate values. And they are local, since the value of the observable $A$ is independent of $\beta^a$, and $B$ is independent of $\alpha^a$.

We assume an isotropic probability density over the space of hidden variables. This density is simply a map $\rho : \Lambda \to [0, 1]$ that assigns equal probability to both cases. With this choice, the observables can easily be shown to yield the right single-side expectation values.

[11] A volume element can be understood to give an orientation to the entire three dimensional manifold in much the same way that antisymmetric rank 2 tensors give an orientation to a plane, by specifying the relative order of any three linearly independent vectors. For more on volume elements, see Malament (2011).

We have for both $A$ and $B$,

$$\mathcal{E}_{h.v.}(A(\alpha^a)) = \frac{1}{2}(\frac{1}{\sqrt{2}}\epsilon_{abc}\alpha^a - \frac{1}{\sqrt{2}}\epsilon_{abc}\alpha^a) = \mathbf{0} \tag{3.1}$$

$$\mathcal{E}_{h.v.}(B(\beta^a)) = -\frac{1}{2}(\frac{1}{\sqrt{2}}\epsilon_{abc}\beta^a - \frac{1}{\sqrt{2}}\epsilon_{abc}\beta^a) = \mathbf{0}. \tag{3.2}$$

Hence, on average, neither body is rotating about *any* direction. This result is consistent with the quantum mechanical expectation values and with experiment.

Finally, we want to consider the joint expectation value. This is where the algebraic properties of rotations become important, since it is at this point that one is trying to establish a relationship between two correlated, rotating bodies. But now consider the product of the outcomes:

$$A_{bc}(\alpha^a, \lambda)B^{bc}(\beta^a, \lambda) = -\frac{\lambda^2}{2}\alpha^n\epsilon_{nbc}\beta_m\epsilon^{mbc} = -\alpha^n\beta_n.$$

The joint expectation value, then, is just

$$\mathcal{E}_{h.v.}(A(\alpha^a), B(\beta^a)) = -\alpha^a\beta_b(\frac{1}{2} + \frac{1}{2}) = -\alpha^a\beta_a, \tag{3.3}$$

again just like in the quantum mechanical case.[12] With this final result, it would appear that we have successfully generated a model of the EPR-Bohm experiment. And more, the joint expectation value associated with this model will violate the Clauser-Horne-Shimony-Holt inequality in exactly the same way as quantum mechanics—after all, it has just the same functional form.

[12]It is at this stage that the model I present diverges most substantially from Christian's. In his proposal, the joint expectation value is calculated using the geometric product of two Clifford algebra-valued observables. In mine, it is the inner product of two antisymmetric rank 2 tensor fields. Antisymmetric rank 2 tensor fields can generally be identified with bivectors in a Clifford algebra, but the products in the two cases are entirely different. It turns out, though, that this distinction just does not matter. Neither product is tracking the right experimental information, and indeed as I will presently show, no such product *could* track the right experimental information and still yield a result that conflicts with Bell-type theorems.

## 4 A diagnosis and a no-go result

What are we to make of the results of sections 2 and 3? One might argue that the results do not actually conflict. Instead, the model just presented reveals that the notion of hidden variable model given by definition 2.1 is too narrow. There, a hidden variable model was defined to have two observables that were integer valued—specifically, valued by elements of the set $\{-1, 1\}$. Rotations in three dimensions, meanwhile, are a complicated business. The correlations exhibited by two rotating bodies cannot be captured in terms of a two element set. One could conclude that it is necessary to consider observables whose possible values reflect the full structure of rotations, as antisymmetric rank 2 tensor do. And indeed, if one does permit such observables, it would appear that one can construct a local, deterministic (generalized) hidden variable model of the EPR-Bohm experiment after all.

On this view, the model just described is not a counterexample to Bell-type theorems *per se*, but it does undermine their foundational importance. Bell-type theorems were of interest because they were understood to exclude a particular, appealing sort of hidden variable theory. If it were to turn out that they only applied to a limited number of those theories—ones with integer-valued observables—the standard moral of Bell-type arguments would be severely muted. There would be hope for finding a local, deterministic hidden variable theory for quantum mechanics after all.

Unfortunately, appealing as this argument might be, it is too fast. The model does not actually work. The problem concerns what, exactly, a (generalized) hidden variable model needs to reproduce. Specifically, Eq. (3.3) does not correctly track the relevant measure of correlation between Alice and Bob's measurement results.

To see the point, consider the following. When Alice and Bob actually run their experiments, they are perfectly entitled to record their measurement results however they choose. They can even record those results as integers $\pm 1$, so long as they do so systematically. (Indeed, this is what experimental physicists actually do.) If a model like the one given above is correct, then what Alice and Bob would be doing would amount to applying a map

$P$ from the codomain of their observables to the set $\{-1, 1\}$. Of course, there is a danger that representing their results in this way could turn out to be a poor choice. One might worry that such a recording mechanism would lose information about their measurement results. If the thing they are observing has more structure than the set $\{-1, 1\}$, then Alice and Bob would risk introducing confusing patterns into their data. These patterns would be recognized as relics of the projection from the actual observables to $\{-1, 1\}$ once the actual observables were known, but without any knowledge of the structure of the observables, the data would be difficult to interpret. Indeed, we already know that the data from EPR-Bohm experiments, standardly represented as sequences of $\pm 1$, do exhibit correlations that are difficult to interpret. But here is the crucial point. Whatever else is the case, the troublesome correlations that are present in EPR-Bohm data arise when the data is recorded as sequences of $\pm 1$. And so it is *these* correlations that a (generalized) hidden variable model needs to reproduce to be satisfactory. In other words, to show that a (generalized) hidden variable model correctly reproduces the EPR-Bohm experimental result, one needs to show that the model recovers the standard quantum mechanical expectation values *after* we project the complicated observables onto $\{-1, 1\}$.

So in order to determine whether the model presented above is satisfactory, we need to find a way for Alice and Bob to systematically project the elements of $\mathbb{R}^3 \wedge \mathbb{R}^3$ onto $\{-1, 1\}$. There is a principled way to do this. It is to take the integer to indicate the orientation of the rotation about the measurement vector (so, $+1$ might record counterclockwise rotation, $-1$ clockwise). In the present case, this representational scheme can be expressed by a map $P : \mathbb{R}^3 \times (\mathbb{R}^3 \wedge \mathbb{R}^3) \to \{-1, 1\}$. The map turns out to be unique (up to sign). For a given unit vector $\xi^a$, and for any antisymmetric rank 2 tensor field $X_{ab}$ in $\mathbb{R}^3 \wedge \mathbb{R}^3$, the map can be defined as,

$$P(\xi^a, X_{ab}) = \frac{1}{\sqrt{2}} X_{ab} \epsilon^{nab} \xi_n.$$

Note that projecting in this way does amount to a loss of structure, insofar as $\{-1, 1\}$ does not have the algebraic properties of rotations. But it is as good a way as any to represent an antisymmetric rank 2 tensor as an integer, which is what we need to do to adequately evaluate the model.

What, then, does the model predict for Alice and Bob's measurements, recorded using $P$? For Alice, for any given choice of measurement vector $\alpha^a$, the answer is given by,

$$P(\alpha^a, A_{ab}(\alpha^a, \lambda)) = \frac{1}{\sqrt{2}}(\frac{\lambda}{\sqrt{2}}\epsilon_{mab}\alpha^m)\epsilon^{nab}\alpha_n = \lambda.$$

For Bob, meanwhile, for any choice of $\beta^a$, we find,

$$P(\beta^a, B_{ab}(\beta^a, \lambda)) = \frac{1}{\sqrt{2}}(-\frac{\lambda}{\sqrt{2}}\epsilon_{mab}\beta^m)\epsilon^{nab}\alpha_n = -\lambda.$$

By definition $\lambda = \pm 1$, so these results are correctly valued in the set $\{-1, 1\}$. Since we have assumed a uniform density function, it follows that

$$\begin{aligned}\mathcal{E}(P(\alpha^a, A_{ab}(\alpha^a)) &= \frac{1}{2} - \frac{1}{2} = 0\\ \mathcal{E}(P(\beta^a, B_{ab}(\beta^a)) &= -\frac{1}{2} + \frac{1}{2} = 0,\end{aligned}$$

which means that Alice and Bob should be getting $+1$ and $-1$ about half the time each. This is in agreement with the quantum mechanical case.

But something is awry. Suppose that Alice and Bob pick their respective vectors $\alpha^a$ and $\beta^a$ and then perform successive measurements with these measurement vectors, recording the results as $\pm 1$ as they go. After many trials, Alice and Bob should have long lists of numbers. For Alice and Bob individually, if you took the average value of each of their results, you would find it tending to 0. This is what it means to say that the individual expectation values are zero. But now, what about the joint expectation value? The joint expectation value is supposed to be a prediction for the average value of the *product* of Alice and Bob's measurement results for each trial. So, if Alice wrote down $+1$ for her first trial, and Bob wrote down $+1$, the product of their outcomes for that trial would be $+1$. If for the next trial, Alice wrote down $+1$ and Bob wrote down $-1$, the product would be $-1$. And so on. The quantum mechanical joint expectation value, $\mathcal{E}_{q.m.}(A, B)$, amounts to a prediction that, over the long run, the sequence of values of $\pm 1$ calculated by multiplying Alice and Bob's individual results for each trial will have an average value of $\alpha^a \beta_a$. This is the prediction that is confirmed by the experimental evidence, that violates the Clauser-Horne-Shimony-Holt inequality, and that would have to be reproduced by a viable local, deterministic (generalized) hidden variable theory. As I have argued, a hidden variable model based on generalized observables would need to reproduce this joint expectation value after projection onto $\{-1, 1\}$ in order to be satisfactory.

Does the model presented in section 3 accomplish this task? Decidedly not. Consider what happens on this model if Alice and Bob choose to record their results using $\pm 1$, via $P$. One immediately sees that for *any* choices of $\alpha^a$ and $\beta^a$, if you take the product of Alice's result and Bob's result for a particular trial, you always get the same number. Specifically,

$$P(\alpha^a, A_{ab}(\alpha^a, \lambda))P(\beta^a, B_{ab}(\beta^a, \lambda)) = -\lambda^2 = -1.$$

In other words the relevant joint expectation value, the joint expectation of the results as represented by integers, is,

$$\mathcal{E}(P(\alpha^a, A_{ab}(\alpha^a)), P(\beta^a, B_{ab}(\beta^a))) = \frac{1}{2}(-1 + -1) = -1.$$

This is not consistent with the quantum mechanical case, nor with experiment. (It also fails to violate the Clauser-Horne-Shimony-Holt inequality.)

As I have already suggested, the problem stems from the joint expectation value defined in Eq. (3.3). There we appeared to show that the joint expectation value for Alice and Bob's measurements was $-\alpha^a \beta_a$, as expected from quantum mechanics. But we now see that we were comparing apples to oranges. The calculation in Eq. (3.3) is a formal trick, something that looks like the right sort of expectation value calculation, but which is actually irrelevant to the problem we were initially interested in. Eq. (3.3) does give *some* measure of the overlap between Alice and Bob's results, but not the measure that we were trying to reproduce.

To make the point more generally, we might begin by defining a new, broader notion of hidden variable model.

**Definition 4.1** *A* generalized hidden variable model *of the EPR-Bohm experiment is an ordered sextuple* $(\Lambda, X, A, B, \rho, P)$*, where* $\Lambda$ *is a space of complete states of the system,* $X$ *is some set of possible measurement outcomes,* $A, B : \mathbb{R}^3 \times \Lambda \to X$ *are maps from detector settings and complete state specifications to measurement outcomes for Alice and Bob respectively,* $\rho : \Lambda \to [0, 1]$ *is a probability density function on the space of complete states, and* $P : \mathbb{R}^3 \times X \to \{-1, 1\}$ *provides a way to represent the elements of* $X$ *as numbers* $\pm 1$*, relative to the choice of detector setting.*

The relevant expectation values can now be seen to be the expectation values of the observables $A$ and $B$ *after* they have been mapped to $\{-1, 1\}$ using $P$. (We will use the symbol $\mathcal{E}_{g.h.v}$ to represent this new notion of expectation value.) We can thus stipulate more clearly that, in order for a generalized hidden variable model $(\Lambda, X, A, B, \rho, P)$ to recover the quantum mechanical expectation values, it must be the case that for every choice of $\mathbf{a}$ and $\mathbf{b}$, the following are satisfied:[13]

$$\mathcal{E}_{g.h.v.}(A(\mathbf{a})) = \int_\Lambda P(\mathbf{a}, A(\mathbf{a}, \lambda))\rho(\lambda)d\lambda = 0 \tag{4.1}$$

$$\mathcal{E}_{g.h.v.}(B(\mathbf{b})) = \int_\Lambda P(\mathbf{b}, B(\mathbf{b}, \lambda))\rho(\lambda)d\lambda = 0 \tag{4.2}$$

$$\mathcal{E}_{g.h.v.}(A(\mathbf{a}), B(\mathbf{b})) = \int_\Lambda P(\mathbf{a}, A(\mathbf{a}, \lambda))P(\mathbf{b}, B(\mathbf{b}, \lambda))\rho(\lambda)d\lambda = -\mathbf{a} \cdot \mathbf{b} \tag{4.3}$$

In these new terms, the model presented in section 3 is indeed a local, deterministic generalized hidden variable model. But it does not reproduce the EPR-Bohm measurement results because it fails to satisfy Eq. (4.3).

These considerations suggest a simple no-go result that might be understood to generalize

[13] I am now switching back to the notation used in section 2, to emphasize the generality of these expressions.

theorem 2.4 to the case of a generalized hidden variable model.[14] Remarkably, the following is an immediate corollary of Theorem 2.4.

**Corollary 4.2** *There does not exist a local, deterministic generalized hidden variable model of the EPR-Bohm experiment that reproduces the quantum mechanical expectation values.*

Proof. Let $(\Lambda, X, A, B, \rho, P)$ be a local, deterministic generalized hidden variable model. Define two new observables $A', B' : \mathbb{R}^3 \times \Lambda \to \{-1, 1\}$ as follows: for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and any $\lambda \in \Lambda$, set $A'(\mathbf{a}, \lambda) = P(\mathbf{a}, A(\mathbf{a}, \lambda))$ and $B'(\mathbf{b}, \lambda) = P(\mathbf{b}, B(\mathbf{b}, \lambda))$. Now $(\Lambda, A', B'\rho)$ is a hidden variable model (in the original sense of definition 2.1). Clearly $(\Lambda, A', B', \rho)$ is local and deterministic just in case $(\Lambda, X, A, B, \rho, P)$ is. Thus, for any $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$, and $\mathbf{b}'$, the expectation values of $A'$ and $B'$ must satisfy the Clauser-Horne-Shimony-Holt inequality. It follows, by direct substitution, that

$$|\mathcal{E}_{g.h.v.}(A(\mathbf{a}), B(\mathbf{b})) - \mathcal{E}_{g.h.v.}(A(\mathbf{a}), B(\mathbf{b}')) + \mathcal{E}_{g.h.v.}(A(\mathbf{a}'), B(\mathbf{b})) + \mathcal{E}_{g.h.v.}(A(\mathbf{a}'), B(\mathbf{b}'))| \leq 2. \tag{4.4}$$

As noted above, the quantum mechanical expectation values do not satisfy this inequality. Thus no local, deterministic generalized hidden variable model can reproduce the quantum mechanical expectation values for an EPR-Bohm experiment. □

It follows that one cannot get around Bell-type theorems by considering more general representations of experimental outcomes.

## 5 Final remarks

The principal moral of this paper is that one cannot get around Bell-type arguments by generalizing the space of measurement outcomes, or equivalently, by representing the state of the particles with more sophisticated mathematical machinery. The reason for this is that ultimately the measurement data that exhibits the Bell-type-theorem-violating correlations

[14] Note that this result applies irrespective of $X$, which means that it precludes models like the one Christian proposes in addition to the model I have discussed.

consists of sequences of elements of the set $\{-1, 1\}$. It is these sequences of the unit integers, or at least their expectation values, that a (generalized) hidden variable model of the EPR-Bohm experiment needs to reproduce in order to be satisfactory. Corollary 4.2 shows how this is impossible, since the combination of any non-integer valued observables with a map that identifies values of the observables with $\pm 1$ simply recovers an observable in the standard sense of Bell-type theorems. And we already know that a hidden variable model with standard, integer-valued observables cannot successfully reproduce the predictions of quantum mechanics.

This last point underscores the scope and generality of Bell-type theorems. The model and the no-go result I present here may help to clarify how it is that Bell-type theorems apply even to models that look quite different than the models Bell describes, and which appear to be the target of standard Bell-type theorems. But the present no-go result is at best a corollary of results that have been known for forty years. This means that Bell-type results do have the foundational importance typically attributed to them, contra the suggestion offered at the beginning of section 3 that Bell made a serious error in how he constrained the representation of EPR-Bohm measurement results.

As a coda to the present discussion, it is interesting to note that if one began with a different version of Bell's theorem, in particular one of the probabilistic variety often associated with the Clauser-Horne inequality and with Jarrett's analysis of Bell-type results, the problems with the model presented in section 3 would have been manifest at an earlier stage. The reason is that one would have been forced to determine whether to consider the probability of arriving at particular measurement results before mapping those results to $\{-1, +1\}$, or after mapping them to $\{-1, +1\}$. In the first case the connection between the predictions of the generalized model and the predictions of quantum mechanics (expressed in the probabilistic language) would have been simply incomparable. And in the second case, it would have been immediately clear that the model was not successfully reproducing the probabilistic predictions of quantum mechanics. But it seems to me that, though rehearsing this argument in detail would have revealed that something was amiss with the model, the origin of the present model's failure would have been more difficult to diagnose.

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