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Quantum particles, individual properties, and discernibility

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Abstract

The paper discusses how to formally represent properties characterizing individual components of quantum systems containing many particles of the same type. It is argued that this can be done using only fully symmetric projection operators. An appropriate interpretation is proposed and scrutinized, and its consequences related to the notion of quantum entanglement and the issue of discernibility and individuality of quantum particles are ascertained.

Introduction

The main purpose of this paper is to consider, in a precise and rigorous manner, the problem of how to formally represent properties of individual particles of the same type, in the case when only the joint state of a group of such particles is given. The quantum theory of many particles imposes the requirement of symmetricity/antisymmetricity on such joint states, and this requirement arguably limits the admissible formal representations of individual properties to symmetric operators. Using specific symmetric operators we can express statements of the form "At least (exactly) one of the particles possesses a given property *P*", but we can never say definitively "Particle number 1 (2) possesses property *P*". Several curious features of the considered interpretation of individual properties will be analyzed, and it will be argued that while they are certainly peculiar, they do not involve any formal or conceptual inconsistency. The adopted representation of individual properties will also be shown to influence our concept of entanglement, leading to a re-evaluation of its standard definition. Finally, we will apply the developed formal apparatus to the perennial metaphysical debate on the discernibility of quantum particles. It will be argued, in contrast to the commonly accepted view, that bosons and fermions of the same type can sometimes be absolutely discerned by their properties.

1. *Preliminary distinctions*

Measurable properties of quantum systems are often represented with the help of projection operators. The image of a projector P is the subspace $P(H)$ of the entire space H such that for every vector $|u\rangle \in P(H)$ there is a vector $|v\rangle \in H$ for which $P|v\rangle = |u\rangle$. Thanks to the fact that P is idempotent (i.e. $P^2 = P$) it can be easily proven that for every $|u\rangle \in P(H)$, $P|u\rangle = |u\rangle$. Thus all vectors in the subspace $P(H)$ onto which P projects are its eigenvectors with the corresponding eigenvalue equal 1. On the other hand, it can also be proven that for any vector $|v\rangle$ orthogonal to the entire subspace $P(H)$, $P|v\rangle = 0$. All this suggests that P can be naturally interpreted as representing the question "Does the state of the system belong to the subspace $P(H)$?" with two possible answers "yes" (represented by value 1) and "no" (value 0). If the expectation value of P in a given state equals 1, this means that this state belongs to the subspace $P(H)$, whereas the value 0 indicates that the state is orthogonal to $P(H)$. All other values indicate states represented by vectors which are "tilted" with respect to $P(H)$.

In the special case when P projects onto a one-dimensional subspace spanned by a vector $|u\rangle$, P can be given in the form of the dyad $|u\rangle\langle u|$, i.e. the operator whose action on any vector $|v\rangle$ gives the vector $\langle u|v\rangle\, |u\rangle$. An expectation value of $|u\rangle\langle u|$ equal to 1 indicates that the state of the system is $|u\rangle$. If $|u\rangle$ is an eigenvector of a particular Hermitian operator A with a corresponding eigenvalue a, the fact that the expectation value of $|u\rangle\langle u|$ is 1 can be interpreted as showing that A possesses a definitive value a. Projection operators are thus flexible enough to represent any measurable and well-defined property of a quantum system.

To sum up: the quantum-mechanical formalism presents us with the following three options regarding the possession of a given property P by a particular quantum system s. System s in state $|\psi\rangle$ possesses the definite property P if and only if $\langle\psi|P|\psi\rangle = 1$. System s in state $|\psi\rangle$ *definitely* does not possess property P iff $\langle\psi|P|\psi\rangle = 0$. And the third case is when $0 < \langle\psi|P|\psi\rangle < 1$, which means that s neither definitely possesses, nor definitely does not possess property P. This is the case of quantum-mechanical indeterminacy, unknown in the classical world.

Quantum-mechanical systems consisting of many components are described using tensor products of one-particle Hilbert spaces. Limiting ourselves to the case of two-particle systems, we may write the state space of such systems as $H_1 \otimes H_2$. In the case when both particles are of the same type, H_1 and H_2 are two copies of the same one-particle Hilbert space. The tensor formalism enables us to represent states which are just ordinary combinations of states for individual particles, but it also creates the possibility of introducing new states not reducible to simple products of the states of separate components. Thus the product vector $|\psi\rangle_1 \otimes |\varphi\rangle_2 \in H_1 \otimes H_2$ obviously represents the situation in which the first particle is in state $|\psi\rangle$ while the second is in state $|\varphi\rangle$, but the superposition $|\psi\rangle_1 \otimes |\varphi\rangle_2 + |\varphi\rangle_1 \otimes |\psi\rangle_2$ lacks such a simple interpretation.

Systems of particles of the same type (two electrons, two photons, etc.) are assumed to obey the Symmetrization Postulate which reduces their admissible states to two types only: symmetric (for bosons) and antisymmetric (for fermions). This is a consequence of the assumption that permuted states should be empirically indistinguishable from each other. Thus no two fermions (bosons) can occupy the state of the form $|\psi\rangle_1 \otimes |\varphi\rangle_2$. The proper way of writing down such a state is to either symmetrize or antisymmetrize it, which leads to state $|\psi\rangle_1 \otimes |\varphi\rangle_2 + |\varphi\rangle_1 \otimes |\psi\rangle_2$ or $|\psi\rangle_1 \otimes |\varphi\rangle_2 - |\varphi\rangle_1 \otimes |\psi\rangle_2$ (throughout the paper I omit normalization constants). In the next step we will try to uncover the consequences of the Symmetrization Postulate related to the problem of how to formally represent the properties of individual components of larger systems of indistinguishable particles.

2. *Systems of indistinguishable particles and their properties*

We will now address the following question: given that a projector P represents a particular property of an individual particle s, how can we represent the fact that the same property is possessed by one of the particles s_1 and s_2 which jointly form a system of two indistinguishable particles? The standard answer to this question is as follows: we take the tensor product $P \otimes I$, where I is the identity operator, to formalize the property P possessed by the first particle. Analogously, the product $I \otimes P$ formalizes the same property possessed by the second particle. But we are facing an immediate problem here. Neither $P \otimes I$ nor $I \otimes P$ is a symmetric operator, which means that their expectation values are not generally preserved under the permutations of particles. In particular, if P is the one-dimensional projector $|u\rangle\langle u|$, then the expectation value of $P \otimes I$ in state $|u\rangle_1 \otimes |v\rangle_2$ is 1, whereas the same operator will have the expectation value equal $\langle v|P|v\rangle$ (which may be zero when $\langle u|v\rangle = 0$) in the permuted state $|v\rangle_1 \otimes |u\rangle_2$. Thus the projector $P \otimes I$ distinguishes between empirically indistinguishable states, and therefore it is doubtful that it should be admitted as a physically meaningful operator.

In order to avoid similar problems we should look for a symmetric projection operator which could represent the statement that one of the two particles possesses a given property P. The crucial thing to observe here is that such an operator cannot definitively decide whether it is particle number 1 or 2 that possesses property P, for if it did, it would not satisfy the requirement of symmetricity. We have to accept the fact that labels which we use in the tensor product formalism do not carry any physical meaning by themselves. We can identify a particle as bearing label 1 (or 2), only if there is a physical property which distinguishes particle 1 from particle 2 (for instance one of them is a negatively charged electron, and the other a positively charged positron). But in the case of two particles of the same type such identification is impossible.

We can now consider the following two candidates for symmetric projection operators capable of expressing the thought that one of the two particles possesses a particular property P:[1]

$$\Omega_P = P \otimes (I - P) + (I - P) \otimes P + P \otimes P$$
$$\Sigma_P = P \otimes (I - P) + (I - P) \otimes P$$

Let $|\lambda_i\rangle$ be any set of vectors spanning the image space $P(H)$ (thus the $|\lambda_i\rangle$ are eigenvectors of P corresponding to the value 1), $|\mu\rangle$ – any vector orthogonal to space $P(H)$ (i.e. an eigenvector of P corresponding to the value 0), and $|\varphi\rangle$ – any vector whatsoever. It can be quickly verified that operators Ω_P and Σ_P give the following results when applied to selected product states of two particles:

$$\Omega_P\, |\lambda_i\rangle_1 \otimes |\varphi\rangle_2 = |\lambda_i\rangle_1 \otimes |\varphi\rangle_2$$
$$\Omega_P\, |\varphi\rangle_1 \otimes |\lambda_i\rangle_2 = |\varphi\rangle_1 \otimes |\lambda_i\rangle_2$$
$$\Omega_P\, |\lambda_i\rangle_1 \otimes |\lambda_j\rangle_2 = |\lambda_i\rangle_1 \otimes |\lambda_j\rangle_2$$
$$\Sigma_P\, |\lambda_i\rangle_1 \otimes |\mu\rangle_2 = |\lambda_i\rangle_1 \otimes |\mu\rangle_2$$
$$\Sigma_P\, |\mu\rangle_1 \otimes |\lambda_i\rangle_2 = |\mu\rangle_1 \otimes |\lambda_i\rangle_2$$
$$\Sigma_P\, |\lambda_i\rangle_1 \otimes |\lambda_j\rangle_2 = 0$$

The interpretation of these facts should be straightforward. Regarding the operator Ω_P we can see that each product vector containing an eigenvector of P as at least one of its components is an eigenvector of Ω_P corresponding to the value 1. Thus it is natural to interpret Ω_P as expressing the fact that at least one particle possesses property P. With respect

[1] I gave a simple formal argument that there may be only two such projection operators elsewhere (Bigaj 2015). Operator Ω_P is used in the extensive analysis of the notion of entanglement in systems of indistinguishable particles done by Ghirardi, Marinatto, and Weber (2002). See also (Ghirardi, Marinatto 2004).

to operator Σ_P we may be tempted to read it as stating that exactly one particle possesses property P. However, this intuitive reading leaves out the fact that for a product of an eigenvector of P with another vector to be an eigenvector of Σ_P, this other vector has to be orthogonal to $P(H)$. Therefore it is better to read the property expressed by Σ_P as "One particle definitely possesses property P while the other definitely does not possess P".

3. *Properties of individual particles in symmetric and antisymmetric states*

We have equipped the projectors Ω_P and Σ_P with appropriate intuitive interpretations on the basis of their actions upon product states. But now we should analyze the behavior of these operators in states that can't be represented as direct products of one-particle states. Such non-factorizable states are usually called "entangled", but we will soon see that this terminology can cause some controversy. Let us start off with a simple case of states which arise as a result of the (anti-)symmetrization of direct products of states. Our starting point will be the product state $|\varphi\rangle_1 \otimes |\psi\rangle_2$, where we assume that $|\varphi\rangle$ and $|\psi\rangle$ are not proportional to each other. In that case the results of its (anti-)symmetrization are straightforward:

(Sym) $\quad |\varphi\rangle_1 \otimes |\psi\rangle_2 + |\psi\rangle_1 \otimes |\varphi\rangle_2$

(Anti-sym) $\quad |\varphi\rangle_1 \otimes |\psi\rangle_2 - |\psi\rangle_1 \otimes |\varphi\rangle_2$

Let us define a one-particle projection operator $P_\varphi = |\varphi\rangle\langle\varphi|$. A quick calculation can convince us that both states (Sym) and (Anti-sym) are eigenvectors of the symmetric operator Ω_P where $P = P_\varphi$. By symmetry the same applies to the operator $P_\psi = |\psi\rangle\langle\psi|$. Thus we can draw the conclusion that in states (Sym) and (Anti-sym) at least one particle possesses property P_φ and at least one particle possesses property P_ψ. (While it is tempting to say more specifically that one particle possesses P_φ whereas the *other one* possesses P_ψ, we should resist this temptation for reasons that will become clear later.) We have to stress not only that we can't tell which of the particles 1 or 2 possesses which property, but that it doesn't even make sense to ask this question, since we haven't defined any operator corresponding to the statement "Particle 1 (2) possesses property P" (such an operator would obviously violate the requirement of symmetricity). This fact only strengthens our earlier observation that we should not attach too much ontological importance to the labels 1 and 2.

The result of the analysis given above can be expressed in the form of the following fact:

(1) If the state of two fermions (bosons) is a result of the antisymmetrization (symmetrization) of a product of two non-identical states, then there is a definitive property *P* represented by a one-dimensional projection operator for which it is certain that at least one fermion (boson) possesses *P*.

It has been proven (see Ghirardi, Marinatto and Weber 2004, p. 75) that the implication in the opposite direction also holds, therefore we have an equivalence between the fact that a given two-particle state $\Psi(1, 2)$ is obtainable by symmetrizing (antisymmetrizing) a product state, and the fact that at least one particle in the joint state $\Psi(1, 2)$ possesses a definite property *P*. Thus, if the joint state of two particles is not a result of symmetrization (antisymmetrization) of a product state, no definite property represented by a one-dimensional projector *P* can be attributed to (at least) one particle.

In the case of two fermions thesis (1) can be strengthened, thanks to the following fact: if the state of two fermions can be written in the antisymmetric form as $|\varphi\rangle_1 \otimes |\psi\rangle_2 - |\psi\rangle_1 \otimes |\varphi\rangle_2$, where $|\varphi\rangle$ and $|\psi\rangle$ are any non-parallel vectors, then we can find a vector $|\varphi_\perp\rangle$ orthogonal to $|\varphi\rangle$ and such that the initial two-particle state can be equivalently rewritten as $|\varphi\rangle_1 \otimes |\varphi_\perp\rangle_2 - |\varphi_\perp\rangle_1 \otimes |\varphi\rangle_2$ (to see this we should only note that vector $|\psi\rangle$ can always be written as a linear combination of $|\varphi\rangle$ and $|\varphi_\perp\rangle$). Now we can easily check that the latter antisymmetric vector is an eigenvector for the operator Σ_P with $P = |\varphi\rangle\langle\varphi|$, which means that it is certain that *exactly one* fermion possesses property *P* while the other one does not possess *P*. However, this is not true for bosons. The vector (Sym) is an eigenstate of Σ_P only if $|\varphi\rangle$ and $|\psi\rangle$ are orthogonal to each other to begin with. Thus if this condition is not satisfied, each particle can show one of the properties P_φ and P_ψ upon measurement.

Due to the above-mentioned facts, fermions of the same type display a curious feature: the information that *at least* one of a pair of fermions possesses a given property *P* suffices to infer that *exactly* one of them will have this property while the other one will not. This feature may be seen as a variant of Pauli's exclusion principle (and it is a direct consequence of the fact that both operators Ω_P and Σ_P have identical expectation values in antisymmetric states, since the symmetric component $P \otimes P$ gives zero in such states). On the other hand, for bosons this entailment does not hold. If the state of two bosons is such that at least one boson has property P, this obviously does not imply that exactly one boson possesses P while the other does not possess P. The lack of such an entailment is to be expected even classically – after all, it is logically incorrect to reason from "at least one" to "exactly one". However, in the quantum case this inference fails for two reasons, one of which has no classical counterpart. First, it is obviously possible that both bosons possess the property P, as the symmetric product state $|\varphi\rangle_1 \otimes |\varphi\rangle_2$ is an admissible bosonic state. But the second possibility is that the state of two bosons may be a result of the symmetrization of a product of two non-orthogonal states. In such a case, even though it is true that at least one boson possesses a given property P, and it is certainly not true that both bosons possess P, we cannot infer that exactly one particle has P while the other one does not have P.

4. *The problem of the "overdetermination" of fermionic states*

It can be proven that fermionic composite systems display an even more unsettling and potentially damaging characteristic regarding their attribution of properties. To begin with, let us assume that two fermions are in the antisymmetric state $|\varphi\rangle_1 \otimes |\psi\rangle_2 - |\psi\rangle_1 \otimes |\varphi\rangle_2$, where $\langle\varphi|\psi\rangle = 0$. As we already know, this state is an eigenstate for symmetric operators Ω_P and Σ_P built out of projectors $P_\varphi = |\varphi\rangle\langle\varphi|$ and $P_\psi = |\psi\rangle\langle\psi|$. But now it can be shown that there are a lot more than only two projectors with this property. If we take any pair of orthogonal vectors $|\lambda\rangle$ and $|\mu\rangle$ lying in the same plane as $|\varphi\rangle$ and $|\psi\rangle$ (i.e. $|\lambda\rangle = a\,|\varphi\rangle + b\,|\psi\rangle$ and $|\mu\rangle = b^*\,|\varphi\rangle - a^*\,|\psi\rangle$ for some coefficients a, b), it can be easily verified that the original antisymmetric vector can be rewritten as $|\lambda\rangle_1 \otimes |\mu\rangle_2 - |\mu\rangle_1 \otimes |\lambda\rangle_2$. This means that this state is also an eigenstate for symmetric projectors Ω_P and Σ_P constructed with the help of the one-particle one-dimensional projectors P_λ and P_μ. Speaking generally, for any one-particle state $|\nu\rangle$ such that $|\nu\rangle$ is a linear combination of $|\varphi\rangle$ and $|\psi\rangle$, the projectors Ω_P and Σ_P, where $P = |\nu\rangle\langle\nu|$, have their expectation values equal 1 in the initial antisymmetric state.

Using the previously accepted interpretation of projectors Ω_P and Σ_P we have to come to the inevitable conclusion that when two fermions are in a state which is obtained from a product of two states by antisymmetrizing it, there is an infinite number of properties P_i represented by one-dimensional non-commuting projectors such that it can be said that exactly one of two fermions possesses P_i. But this looks like a paradox. Try as we might to distribute these incompatible properties P_i over the two particles, we apparently end up with a bunch of incompatible properties attributed to one and the same object. But we know that projectors P_i do not share eigenstates (they are operators projecting onto distinct one-dimensional subspaces of the one-particle Hilbert space). Thus no particle, taken separately, can be in a state which is an eigenstate for all these operators.

However, we have to be careful here not to jump to conclusions. The problem of overdetermination, as I suggest we call it,[2] does not involve any mathematical impossibility or contradiction. While one-particle, one-dimensional projection operators P_i do not have common eigenvectors, generally it may happen that incompatible observables share some eigenvectors, especially when there is degeneracy involved. And symmetric projectors of the type Ω_P or Σ_P are affected by degeneracy, since they project onto more than one-dimensional subspaces. Hence these subspaces may have common elements. We can illustrate this using a simple case of two spin-1/2 particles and considering only their internal degrees of freedom. In this case the state space is the tensor product of two identical 2-dimensional Hilbert spaces $H^2 \otimes H^2$. Let us now consider two incompatible one-particle projection operators $P_x = |\uparrow_x\rangle\langle\uparrow_x|$ and $P_z = |\uparrow_z\rangle\langle\uparrow_z|$. The projector operator Ω_P, where $P = P_x$, projects onto the three-dimensional subspace of $H^2 \otimes H^2$ spanned by the vectors $|\uparrow_x\rangle_1 \otimes |\downarrow_x\rangle_2$, $|\downarrow_x\rangle_1 \otimes |\uparrow_x\rangle_2$, $|\uparrow_x\rangle_1 \otimes |\uparrow_x\rangle_2$, while in the case of the projector P_z the analogous space is spanned by $|\uparrow_z\rangle_1 \otimes |\downarrow_z\rangle_2$, $|\downarrow_z\rangle_1 \otimes |\uparrow_z\rangle_2$, $|\uparrow_z\rangle_1 \otimes |\uparrow_z\rangle_2$. It can be easily checked that the two subspaces $\Omega_{P_x}(H^2 \otimes H^2)$ and $\Omega_{P_z}(H^2 \otimes H^2)$ have a non-zero intersection, and their common element is exactly the one-dimensional subspace of anti-symmetric vectors $|\uparrow_x\rangle_1 \otimes |\downarrow_x\rangle_2 - |\downarrow_x\rangle_1 \otimes |\uparrow_x\rangle_2 = |\uparrow_z\rangle_1 \otimes |\downarrow_z\rangle_2 - |\downarrow_z\rangle_1 \otimes |\uparrow_z\rangle_2$.

However, there remains a slight inconvenience at the level of interpretation. It certainly sounds strange to admit that when two fermions are in the antisymmetric state $|\uparrow_x\rangle_1 \otimes |\downarrow_x\rangle_2 - |\downarrow_x\rangle_1 \otimes |\uparrow_x\rangle_2$, it is true that whatever direction in space we select, both particles will have definite and opposite values of spin in this direction. Such a situation never occurs when we consider states of individual particles taken separately: for each fermion there is at most one direction in which its value of spin can be well-defined. We may be tempted to explain this unusual situation by classifying it as yet another strange result of entanglement, since the considered antisymmetric state of two fermions clearly looks identical to the well-known entangled EPR state. However, this would be a strange manifestation of entanglement indeed. We are used to the idea that entanglement gives rise to new, surprising phenomena (such as non-local correlations, or non-separability of individual states) which depart radically from

[2] Ghirardi, Marinatto and Weber call it "the problem of arbitrariness" (2004, pp. 77-78 and 84-86).

our classical intuitions regarding the world. Here, on the other hand, the situation is exactly opposite: the particles behave in a sense more classically, since they are attributed more properties than we usually admit in quantum mechanics.

Another point worth mentioning is that the attribution of incompatible properties to elements of composite fermionic systems in no way leads to experimental predictions which would contradict the statistics known to be satisfied by quantum particles. The key issue here is that we don't know which particle possesses which of the two definite values for spin in a given direction. Thus when we randomly select one of the two fermions prepared in the antisymmetric, EPR-like state, the probability that the result of measurement of any spin component is "up" is still ½.

Taking into account the above-stated facts, we may come to the conclusion that the state identified earlier as the EPR spin-state does not deserve to be classified as entangled. Indeed, this is what Ghirardi, Marinatto and Weber urge.[3] They define their notion of entanglement (which we can call GMW-entanglement to differentiate it from entanglement understood as non-factorizability) in such a way that all eigenstates of the operator Σ_P, where P is a one-dimensional projector, are categorized as non-entangled. Thus all states which are the results of the symmetrization/antisymmetrization of two orthogonal states are not GMW-entangled. The EPR state turns out to be GMW-entangled only when we take into account the spatial degrees of freedom. When we use kets $|R\rangle$ and $|L\rangle$ to symbolize two separate spatial locations, the standard way of writing down the EPR state with respect to spins would be

(EPR) $$|\uparrow_x\rangle_1|R\rangle_1 \otimes |\downarrow_x\rangle_2|L\rangle_2 - |\downarrow_x\rangle_1|R\rangle_1 \otimes |\uparrow_x\rangle_2|L\rangle_2,$$

and, after properly antisymmetrizing this state, we should arrive at the following expression:

(EPR-Anti) $$(|\uparrow_x\rangle_1 \otimes |\downarrow_x\rangle_2 - |\downarrow_x\rangle_1 \otimes |\uparrow_x\rangle_2)\ (|R\rangle_1 \otimes |L\rangle_2 + |L\rangle_1 \otimes |R\rangle_2).$$

It can be verified that the state (EPR-Anti) is not an eigenstate of any operator Σ_P with P projecting onto a one-dimensional subspace, and therefore (EPR-Anti) is classified as GMW-entangled.

[3] A similar claim is defended in Ladyman, Linnebo and Bigaj (2013).

5. *Discernibility of particles of the same type*

The philosophical literature on the problem of discernibility of fermions and bosons of the same type is enormous (see the vast bibliography given in French 2011). Virtually all authors writing on this subject accept the Indiscernibility Thesis which states that fermions (bosons) of the same type are not discernible from each other by their properties (this type of discernibility is known as absolute). Thus it is commonly accepted that the Leibnizian Principle of the Identity of Indiscernibles fails for particles obeying quantum statistics. This fact raises a further question of whether elementary particles are genuine individual objects equipped with their own identities. Two general answers to this question are extensively explored: one admits that quantum particles are non-individuals in an appropriate sense of the word which needs to be further explicated, while the other strategy is to try to find an alternative way of grounding individualities of quantum objects other than their absolute discernibility (for instance using weak discernibility by symmetric and irreflexive relations, as explained in Saunders 2006, Muller and Saunders 2008, Muller and Seevinck 2009). In what follows I will argue that perhaps the whole debate started on the wrong foot, since the Indiscernibility Thesis is not as ironclad as it might seem at first sight.

The main argument used to support the statement that particles in symmetric/antisymmetric states are indeed not discernible by their properties assumes that properties of individual components of composite systems of particles are represented by non-symmetric tensor products of operators of the kind $P \otimes I$ and $I \otimes P$. It is straightforward to observe that the expectation values of such operators in symmetric and asymmetric states are identical: $\langle\psi|P \otimes I|\psi\rangle = \langle P_{12}\psi|P \otimes I|P_{12}\psi\rangle = \langle\psi|P_{12}\, P \otimes I\, P_{12}|\psi\rangle = \langle\psi|I \otimes P|\psi\rangle$, where P_{12} is the permutation operator. However, as I indicated at the beginning of section 2, it is doubtful that operators $P \otimes I$ and $I \otimes P$ should be admitted as legitimate representations of properties of individual components of composite systems consisting of particles of the same type. In keeping with the postulate of symmetricity and the rule of attaching no physical importance to labels, we should use an alternative quantum-mechanical representation of properties of one-particle systems. And indeed the analysis done so far can afford us a simple and useful tool to present the problem of absolute discernibility of particles. This tool is just our projector operator Σ_P, whose intuitive meaning was "one particle definitely possesses property P while the other definitely does not possess property P". The situation when one object has a property that the other lacks clearly deserves to be called "discerning", hence we can adopt the following quantum-mechanical interpretation of discernibility:

(2) Two particles of the same type forming a composite system S in state Ψ are absolutely and categorically discerned by property P (represented by a one-dimensional projector) iff the expectation value of Σ_P in Ψ equals 1.

The term "categorically" indicates that the discernment is done with the help of precise values of some complete, non-degenerate observable. Now we can appeal to the facts, stated earlier, which concern the situations when Σ_P admits the expectation value equal 1. We have seen that when the state of two fermions or bosons is a result of (anti-)symmetrization of a product of two orthogonal states, the expectation value of Σ_P is indeed 1 for some selected projectors P. Thus in these states the particles can be said to be absolutely and categorically discerned by their properties. In the case of fermions we don't even have to assume the orthogonality of the initial states, thanks to the fact proven in section 3. However, for bosons the assumption of orthogonality is essential, since in symmetric states arising from products of non-orthogonal vectors the expectation value of Σ_P is less than 1. Summing up, being in a composite state obtainable by symmetrizing/antisymmetrizing a product of two orthogonal states is necessary and sufficient for the particles to be absolutely and categorically discerned by some properties represented by one-dimensional projection operators.

We may further ask the question whether it is possible to relax a bit the condition of discernibility to include projection operators of dimensionality greater than one. The idea is that if there is a one-particle projection operator P projecting onto a higher-dimensional subspace of H, such that the two-particle operator Σ_P receives the expectation value 1, it still may be claimed that the two particles are discerned by P, albeit not categorically. It may be even surmised that in some antisymmetric/symmetric states which cannot be given as a result of antisymmetrization/symmetrization of a product state, the particles could still be discerned by high dimensional projection operators. This conjecture turns out to be correct, as seen in the following fact.

It can be proven generally that two particles can be absolutely discerned by properties represented by projectors P of any dimensionality k iff their state can be written as follows

(3) $\sum_{j=k+1}^{n}\sum_{i=1}^{k} c_{ij}(|\Phi_i\rangle|\Phi_j\rangle \pm |\Phi_j\rangle|\Phi_i\rangle)$

where $|\Phi_1\rangle, \ldots, |\Phi_k\rangle$ are mutually orthogonal vectors spanning $P(H)$, and $|\Phi_{k+1}\rangle, \ldots, |\Phi_n\rangle$ – orthogonal vectors spanning the complement of $P(H)$. As an illustration, let us make the following identifications: $\Phi_1 = |\uparrow\rangle|R\rangle$, $\Phi_2 = |\uparrow\rangle|L\rangle$, $\Phi_3 = |\downarrow\rangle|L\rangle$, $\Phi_4 = |\downarrow\rangle|R\rangle$. It can be verified that the state (EPR-Anti) can be represented in the form of (3), where $n = 4$, $k = 2$, $c_{13} = c_{24} = 1$, $c_{23} = c_{14} = 0$. Thus particles in the EPR-Anti state (which is GMW-entangled) are discerned by the property associated with the subspace spanned by Φ_1 and Φ_2 ("has spin $\uparrow$ and is located either in R or in L").

We should conclude by noting that the admissible states of two bosons or two fermions can be divided into the following categories depending on whether particles in these states can be claimed to be discerned by their properties:

State	Bosons	Fermions
Product of two identical states	Not discernible	Not applicable
Obtainable by (anti-) symmetrizing two orthogonal states	Discernible by one-dimensional projectors	Discernible by one-dimensional projectors
Obtainable by (anti-) symmetrizing two non-orthogonal states	Not discernible	Discernible by one-dimensional projectors
Expressible in the form of (3) for $k > 1$	Discernible by many-dimensional projectors	Discernible by many-dimensional projectors
All other states	Not discernible	Not discernible

Thus, contrary to the common wisdom, bosons and fermions are absolutely discernible in various symmetric (antisymmetric) states.

Bibliography

Bigaj, T. (2015). Exchanging quantum particles. *Philosophia Scientiae 19(1)*. Available on-line: https://www.academia.edu/3747814/Exchanging_quantum_particles.

French, S. (2011). Identity and individuality in quantum theory. *The Stanford Encyclopedia of Philosophy (Summer 2011 Edition)*, E. Zalta (ed.), URL = <http://plato.stanford.edu/archives/sum2011/entries/qt-idind/>.

Ghirardi, G., & Marinatto, L. (2004). General criterion for the entanglement of two indistinguishable particles. *Physical Review , A 70*, 012109.

Ghirardi, G., Marinatto, L., & Weber, T. (2002). Entanglement and properties of composite quantum systems: a conceptual and mathematical analysis. *Journal of Statistical Physics , 108* (112), 49-122.

Ladyman, J., Linnebo, O. & Bigaj, T. (2013). Entanglement and non-factorizability. *Studies in History and Philosophy of Modern Physics*, 43 (3), 215-221.

Muller, F., & Saunders, S. (2008). Discerning fermions. *British Journal for the Philosophy of Science , 59*, 499-548.

Muller, F., & Seevinck, M. (2009). Discerning elementary particles. *Philosophy of Science , 76*, 179-200.

Saunders, S. (2006). Are quantum particles objects? *Analysis , 66*, 52-63.