Fluctuations of the Electromagnetic Vacuum Field or radiation reaction?

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Abstract

The fact that various physical effects usually ascribed to vacuum fluctuations can also be accounted for by the radiation reaction field suggests that perhaps, there is little evidence for vacuum field fluctuations. But is that so? Where does the underdetermination between vacuum field and radiation reaction field come from, and what can be proposed to lift it? What other evidence do we have that the vacuum field is at least partly responsible for these effects? Furthermore, vacuum fluctuations have been ascribed to the Uncertainty Relations (UR). To what extent are such claims justified, and what interpretation of the UR do vacuum fluctuations suggest?

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1 Introduction

1.1 The issues

Whether zero-point fluctuations of the electromagnetic field are physically real has been the issue of debate among physicists in the 20th century. Although opinion is now overwhelmingly in their favor, the strength of the evidence on which this opinion relies is still occasionally challenged,[1] and whether some of the arguments against it still have import is not clear a priori. Furthermore, the issues raised in these debates deepen our understanding of the electromagnetic vacuum field.
Evidence for the existence of vacuum fluctuations has naturally been sought in observable physical phenomena thought to involve them. Yet it has been shown that most of these, including the famous Casimir effect, can also be accounted for without invoking vacuum fluctuations, but instead the field radiated by the charge itself — usually called either the "radiation reaction field" or the "source field" in this context. Of great interest is the fact that these two alternatives (i.e. vacuum vs source field) do not constitute completely unrelated models. It is clear how one can switch from one physical picture to the other by a specific change in formalism: depending on how one chooses to order the operator relating to the system and the operator relating to the electromagnetic field, one finds the vacuum field and the radiation reaction field having different contributions to the effect.
At the same time, the alternative interpretations in terms of vacuum or source field have often been thought to be intimately related to the Fluctuation Dissipation Theorem (FDT).
Are the alternatives pictures brought about by the choice of ordering an instance of theory underdetermination, or can we find a fact of the matter regarding which one holds? What arguments and criteria have been used in support of these alternatives? What part has the FDT played in these debates? How has it been interpreted, and what can it really teach us about vacuum fluctuations? And having surveyed these debates, where can we say the matter stands now? What seems to be the best evidence in favor of vacuum fluctuations, currently?

In order to investigate these questions, this paper is organized as follows.
The rest of this introductory section is devoted to preliminary clarifications: we shall first discuss what is meant by vacuum field, zero-point energies and vacuum fluctuations, then we shall survey what physical effects have been considered as evidence for vacuum fluctuations. As these have also been interpreted in terms of the radiation reaction field, we shall then briefly examine the latter.
The second section discusses how the ordering of operators referring to the electromagnetic field and atomic variables leads to underdetermination in the interpretation of the physical effects previously surveyed: are they due to vacuum fluctuations, the source field, or both? We shall present the results obtained by the physicists who have aimed to settle the matter by arguing that, despite

[1] See notably [1]

leading to the same experimental predictions, one ordering is superior to the others.
The third section is devoted to the Fluctuation Dissipation Theorem (FDT). While its meaning has always seemed cleared in the classical, high temperature limit, we shall see that it is not so in the $T = 0$ limit that involves zero-point energies. The way it has been interpreted in this context, and the corresponding implications for the issue at hand will then be examined in a survey of the relevant physics literature. Most physicists have argued that it provides evidence for vacuum fluctuations, but some have doubted it, and Edwin Jaynes in particular stressed at some point an entirely different view.
The Fourth section will present an argument for vacuum field effects. It shows that the quantum theory of radiation would be inconsistent if it were not for the vacuum field, and for the fact that it is related to the radiation reaction force in a specific way. This argument has been related to the FDT, but is in fact based on the commutation relation between the position and momentum of a particle.
Finally, the paper will end with a discussion of the evidence the previous sections provide in favor or against the existence of vacuum fluctuations. It shall also examine how the latter relate to the uncertainty relations (UR), and what interpretation of the latter it would suggest.

1.2 Vacuum fluctuations, virtual particles and zero-point energies: what are they?

The concept of vacuum fluctuations arises in Quantum Field Theory (QFT), in connection to the zero-point energy of the electromagnetic field. The notion of virtual particles is often found associated to it, and both are often stated to arise courtesy of the uncertainty relations (UR).

1.2.1 Quantum Field Theory (QFT) vs Classical Field Theory and Non-Relativistic Quantum Mechanics (NRQM)

The basic premise of QFT is that matter and radiation can be described in terms of quantum fields.
The difference between the quantum fields of QFT and their classical counterparts is that the former are "quantized", in a sense analogous to the way the energy of a particle is quantized in non-relativistic quantum mechanics (NRQM).
The main difference between QFT and NRQM lies in the fact that the basic entities of interest in QFT are fields, while the particles that NRQM takes as its direct focus are derivative concepts in QFT, in the following sense. A field is by definition an entity that extends throughout space and whose value generally differs between points.[2] In QFT a field is defined for each type of particle one wishes to deal with (there is a "photon field", a "fermion field" for electrons, one for positrons, etc...) These being the basic entities of interest, states in QFT

[2]And generally between events, i.e. points in spacetime.

are states *of a field.* The notion of particle arises in this context as a property of the field: the state of a field is a "multiparticle state", which specifies how many particles have, say, a momentum of a given value when the field is in that state.[3].

In this description, particles are localized energy quanta, that can be thought of as "local excitations" of the corresponding field. For instance, a photon of frequency ω is an excitation of the photon field of energy $\hbar\omega$. Now talking of particles as field excitations certainly seems to imply that the field is thought to be there anyway, even if no excitations happen to be. And indeed such a situation is thought of as a state of the field: its so-called "vacuum state". This state is the ground state of the field in the same sense as a NRQM particle has a ground state: it is its state of lowest energy.[4]

1.2.2 Zero-point energies and vacuum state

Recall that in NRQM a particle in a "harmonic oscillator" potential energy well can only have energies of $(n+\frac{1}{2}\hbar\omega)$.[5] This notably implies that its minimum energy is not zero, but $(\frac{1}{2}\hbar\omega)$, which is referred to as its *zero-point energy.* This is often described as a consequence of the uncertainty relations (thereafter "UR") between position and momentum, which imply corresponding uncertainties for the potential and kinetic energies, respectively. What motivates this view will be discussed below, in section 5, together with other issues pertaining to the UR.

Just as there are position and momentum commutation rules in NRQM, in QFT there are commutation relations between the quantum field and the canonical momentum.[6] The latter in turn result in commutation relations between field

[3] Other properties of interest are energy, spin or polarization

[4] This is ignoring many serious issues with a particle interpretation of QFTs. In reality the validity of a particle description in QFT has been vigorously debated in the philosophical literature. There are a number of concerns, associated with different requirements one would expect particles to satisfy, notably countability or localizability. When demanding that particles be countable and obey relativistic energy conditions, we do have grounds for thinking that a particle interpretation for the Fock representation of *free* fields is possible. The eigenvectors of the total number operator have properties appropriate for states containing definite numbers of particles. The no-particle, vacuum state being invariant under the Poincaré Group, it looks the same to all inertial observers, and most importantly, the energy eigenvalues of one- and multi-particle states are $n\sqrt{\mathbf{k}^2+m^2}$ in agreement with relativity. However this no longer obtains for interacting fields. There are serious problems, both when one tries to represent the interacting field using the free field Fock representation (no state can be interpreted as containing zero quanta, for reasons related to Haag's theorem), and when seeking instead a different Hilbert space representation (attempts involve expressions that are not relativistically covariant or no longer ensure that one-particle states have the desired relativistic energy). Furthermore, even without taking interactions into account, the Unruh effect, whereby an observer accelerating in vacuum would detect Rindler quanta, also presents a challenge for countability.[2], [3], [4], [5]. Localizability also faces problems of its own. The fact that under a set of reasonable conditions, the probability of finding a particle in any spatial set is zero constitutes a no-go theorem, and this conclusion has been generalized to generic spacetimes, as well as for unsharp localization. [6], [7]

[5] i.e. whose potential energy goes as the square of the position coordinate.

[6] Just as there is a correspondence between the Poisson brackets of classical *particle* theories and the commutators of NRQM, there is also a correspondence between Poisson brackets

creation and annihilation operators.[7] As in NRQM, the non-commuting character of these operators again lead to a non-zero eigenvalue for the ground state, i.e. the so-called "vacuum state" — or simply "the vacuum". And again, we find an energy of $\frac{1}{2}\hbar\omega$ involved. However whereas in NRQM this value represented the energy of a particle, now in QFT it represents an energy at each point in space. Another difference from NRQM is that this "zero-point energy of the field" does not simply involve one value of ω, but is a sum over a range of values of ω, which a priori could run over arbitrarily large values. This is what leads to the energy of the vacuum state being infinite.[8] These energies of $\frac{1}{2}\hbar\omega$ are considered half quanta since in QFT as in NRQM energy quanta are $\hbar\omega$.[9] For a particle to be present takes a field excitation of a full quantum, so the vacuum state represents a state where no particles are present, thereby justifying the term "vacuum".
However we often find the claim that it is seething with "virtual particles", associated with the half quanta. The phrase "virtual particles" is often used interchangeably with that of "vacuum fluctuations". In the context of the vac-

involving classical *field* theory variables and the canonical commutation relations of QFT.

[7]As we also find commutation relations between raising and lowering operators in NRQM

[8] The hamiltonian for a free scalar field turns out to be given by:

$$H = \sum_k \frac{1}{2}\,\omega_k\ (a(\vec{k})\,a^\dagger(\vec{k}) + a^\dagger(\vec{k})\,a(\vec{k})), \tag{1}$$

for the particles, with similar terms for the anti-particles. Now if $a(\vec{k})$ and $a^\dagger(\vec{k})$ commuted, by definition that would mean that the two terms $a(\vec{k})\,a^\dagger(\vec{k})$ and $a^\dagger(\vec{k})\,a(\vec{k})$ were equal. It can be shown that $a^\dagger(\vec{k})\,a(\vec{k})$ is a number operator, $N(\vec{k})$, i.e. when it acts on a state it returns the number of particles of momentum $(\vec{k})$ present in that state (analogously to the NRQM number operator returning the number of energy quanta of a particle). We would then get:

$$a(\vec{k})\,a^\dagger(\vec{k}) = a^\dagger(\vec{k})\,a(\vec{k}) = N(\vec{k}), \tag{2}$$

and when H acts on the vacuum field we would find:

$$H|0\rangle = \sum_k \omega_k\ N(\vec{k})|0\rangle = \sum_k \omega_k\ .0|0\rangle = 0, \tag{3}$$

i.e., the vacuum state would have an energy eigenvalue of zero.
This is not the case because $a(\vec{k})$ and $a^\dagger(\vec{k})$ do not commute, and instead $[a(\vec{k}), a^\dagger(\vec{k})] = 1$. Then what we get is:

$$a(\vec{k})\,a^\dagger(\vec{k}) = a^\dagger(\vec{k})\,a(\vec{k}) + 1 = N(\vec{k}) + 1 \tag{4}$$

$$a(\vec{k})\,a^\dagger(\vec{k}) + a^\dagger(\vec{k})\,a(\vec{k}) = 2a^\dagger(\vec{k})\,a(\vec{k}) + 1 = 2N(\vec{k}) + 1, \tag{5}$$

and when H acts on the vacuum field we find:

$$H|0\rangle = \sum_k \omega_k\ \left(N(\vec{k}) + \frac{1}{2}\right)|0\rangle = \sum_k \omega_k \frac{1}{2}|0\rangle. \tag{6}$$

With the sum running over an infinite number of values of k, this sum is infinite, and we find that the energy of the vacuum state is infinite.
The issue is often dealt with by imposing that the operators should be "normal ordered", that is ordered in such a way that the annihilation operator is always to the right of the creation operator. Imposing this to Eq.(1) yields Eq.(3). However what we are essentially doing then is arbitrarily imposing that $a(\vec{k})$ and $a^\dagger(\vec{k})$ commute.

[9]The hamiltonian still involves $\hbar\omega N$, not half thereof.

uum state, “virtual particles” refers to the idea that the UR between energy and time, $\Delta E \Delta t \geq \frac{\hbar}{2}$, allows the field to have an excitation of a full quantum of energy for a very brief time.[10] This time is thought to be too short *as a matter of principle* for the particles to be observed, as a consequence of the UR. What part exactly the half quanta play in this account is often not made explicit, but they thought to arise courtesy of the UR by analogy to the ground state energy of the NRQM harmonic oscillator.[11]
Whether sense can be made of this scheme, and what implications it would have for our understanding of the UR, is discussed below in section 5.
Finally, let us already mention an operational definition of the electromagnetic vacuum field that we shall encounter later. In some of the research on the topic, the concept of “free field” is used instead, by which is meant “the solution of the homogeneous field equation (without atomic source term), [which] coincides with the “vacuum field” when no [real] photons are initially present”.[12]
Also, the way in which the vacuum is identified as such in derivations is often through its spectral energy density $\rho_0(\omega)$, that is its energy density not only per unit volume, but also per frequency interval $d\omega$. This quantity is the product of the energy of one mode — which in the case of the vacuum is half a quanta, $\frac{1}{2}\hbar\omega$, and the density of modes per unit volume.[13] The resulting expression is:

$$\rho_0(\omega) = \frac{\hbar\omega^3}{2\pi^2 c^3}.^{14} \tag{7}$$

1.3 Physical effects deemed evidence for vacuum fluctuations

The physical effects attributed to vacuum fluctuations include the Lamb shift, spontaneous emission and the Casimir force. However, they can also be accounted for by radiation reaction.
The Lamb shift consists in a deviation from the prediction of non-relativistic quantum mechanics regarding the spectrum of the hydrogen atom, according to which the $2s_{1/2}$ and the $2p_{1/2}$ states[15] should have the same energy. The Lamb shift is usually interpreted as arising from fluctuations in the position of the electron within the atomic potential. These fluctuations themselves are attributed to interactions between the electron and the electromagnetic field vacuum fluctuations.[16] But they can also be thought to arise instead from the electromagnetic field associated with radiation reaction.
Spontaneous emission is commonly interpreted as a form of *stimulated* emis-

[10]Hence the notion of “fluctuations”, i.e. variations in time.
[11]Particle in an harmonic oscillator potential well.
[12][8], p.1618. This was standard use, and can notably be found in [9].
[13]Also sometimes referred to as density of states.
[14]The density of modes is $\frac{\omega^2 d\omega}{\pi^2 c^3}$.
[15]principal quantum number $n = 2$, orbital quantum number $l = 0$ and 1 respectively.
[16][10], pp.635-636; [11], pp.61-62.

sion, with vacuum fluctuations as the origin of the disturbance.[17] Prior to this explanation however, the source of the disturbance used to be ascribed to the radiation reaction field of the electron ([12], pp.142-143).
The Casimir force is an attractive force between neutral conducting plates. It is traditionally ascribed to the energy density of the vacuum fluctuations between the plates being smaller than outside of them: this difference in energy densities decreases if the distance between the plates is reduced. There does not seem to be much room for radiation reaction in this picture, however the Casimir force can also be derived by considering the polarization energy of the atoms in the plates, and how this energy changes as the distance between the plates is varied. This polarization can be attributed to the vacuum field, but also to the radiation reaction field.
As will be discussed shortly, which field is responsible can be shown to depend on the way one orders the operators for the field and the atom.

1.4 Source field and radiation reaction: what are they?

Unlike the vacuum electromagnetic field, radiation reaction is a classical concept. The radiation reaction force, also known as the Abraham-Lorentz force, is the recoil force on a radiating charge due to its own radiation, just as the thrust is the recoil force on a rocket due to the exhaust gases it ejects, or the recoil is the force on a gun when it shoots a bullet. In all these situations, energy and momentum conservation does not apply to the more massive system of interest (charge, rocket, gun), but to the larger system that includes what is being ejected (electromagnetic radiation, exhaust gases or bullet, respectively). As the latter carry away part of the momentum of the total system, the momentum of the system of interest also changes — so that the total momentum remains the same. This implies that a force is exerted on the system of interest, by the ejected part. So for a radiating charge, the recoil, "radiation reaction" force, is being exerted on the charge by the very radiation it emits. This is why the relevant field is referred to as the "source field": it is emitted by a source, i.e. the charge, by opposition to the vacuum field.
The radiation reaction force is proportional to $\dddot{x}$, the time derivative of the acceleration of the charge:

$$F = -\frac{2}{3}\frac{e^2}{c^3}\,\dddot{x} \tag{8}$$

where e is the charge, c the speed of light and x the position of the charge.[18]

[17] Spontaneous emission is the emission of a photon by an atom, during which the atom transitions from a higher to a lower excited state, without the process being due to the atom having been visibly disturbed.

[18] [13], chapter 28, pp.6-7. There are interesting issues regarding the radiation reaction force, which we shall not dwell on here but are at least worth mentioning. It accounts for the energy a charge loses when radiating, however what it conserves is the *on average* loss, over an entire period of oscillation of an emitting dipole for instance, not the instantaneous one.

2 Operator ordering and ambiguity in the physical interpretation

As discussed above, the physical effects attributed to the vacuum field (Lamb shift, spontaneous emission, the Casimir force) can also be attributed to radiation reaction, that is to a source field. Which of these two fields is responsible has be shown to depend on the way one orders the operators for the field and the atom.

2.1 Operator ordering as the source of underdetermination

In the present context, the ambiguity in the physical picture that the formalism depicts originates from the fact that the quantities of interest involve the product of two operators, and that a priori we are free to order these in any way we wish. To see this let us represent by Q the operator for the quantity that we wish to find. In the simplest case Q is proportional to the product of an operator for the total electromagnetic field, E, with an operator that refers to some atomic variable, which we shall call P. That is:

$$Q \sim PE. \tag{9}$$

E and P commute at equal times, so we are free to order them any way we wish: the derivations that follow yield the same results.[19]
However different choices of ordering lead to different physical pictures because E is the sum of operators that do *not* commute with P — the 0-point vacuum field E_0 and the source field E_S:

$$E = E_0 + E_S, \tag{10}$$

The quantity of interest, Q, is due to the two corresponding contributions, Q_0 and Q_S.[20] If we chose the order P E, we now find:

$$Q = PE = P(E_0 + E_S) = PE_0 + PE_S = Q_0 + Q_S \tag{11}$$

And for E P:

$$Q = EP = (E_0 + E_S)P = E_0P + E_SP = Q_0 + Q_S \tag{12}$$

[19] This is discussed by Dalibard *et al* before they treat specifically of the Lamb shift and the spin anomaly of the electron.[8] Because of the application they have mind, they consider the time evolution of an operator G, $\frac{dG(t)}{dt}$, rather than an arbitrary operator Q as I have done here.

[20] [8] refers to these quantities as $\left(\frac{dG(t)}{dt}\right)_{vf}$ and $\left(\frac{dG(t)}{dt}\right)_{sr}$, respectively, where the subscripts stand for "vacuum field" and "self-reaction".

That seems all fine, but although their sum does, E_0 and E_S do not commute with P:

$$E_0 P \neq \; PE_0 \quad E_S P \neq \; PE_S.^{21} \tag{13}$$

So Q_0 is not the same thing in Eq.(11) as in Eq.(12) — and the same goes for Q_S of course. Therefore, depending on how we choose to order P and E, the total effect represented by Q will be due to contributions from Q_0 and Q_S in different proportions — even though one recovers the same result for Q itself whatever choice is made.

We see that the underdetermination arises because, in so far that some operators commute, we get similar final results, in so far that the operators that make them up do not commute, we get different accounts of which field is responsible for the effects.

2.2 Reactions to the ambiguity

Different attitudes towards this underdetermination can be found in the literature. There is agreement regarding the formal equivalence of the choice of ordering. However some researchers have argued that physical interpretation itself should be ground for regarding one ordering as being the correct one. Specifically, they deem that each separate contribution (i.e. such as Q_0 and Q_S above) should have a clear physical meaning, and that for this to be the case, they need to be formally described by a Hermitian operator. Only symmetric ordering corresponds to this case.[22]

Therefore, at this point it would seem that imposing the condition that Q_0 and Q_S must be Hermitian implies symmetric ordering.

This is not true in all cases however, notably not when dealing with a model of relevance to our concerns: the two-level atom.[23]

Indeed in this case the operator of interest Q is given not by PE but by a sum over terms of the form $E^- P^- + E^+ P^+$.[24] In this case choices of ordering that lead to different relative contributions from E_0 and E_S nevertheless correspond

[21] An explanation given for this is:

> One's freedom to change orderings in mid-calculation is limited because different operators may acquire differing time arguments and because the interaction of the field with the atom may make the separation of different degrees of freedom difficult at a time other than the initial instant. [9], p.157.

It may seem strange that operators referring to a field and an atom, hence to different systems, can fail to commute. However, if we consider for instance the polarization of an atom, the fact that it is due to the field leads to the polarization being expressed in terms of the creation and annihilation operators of the field.

[22] [8]. See the Appendix for the derivation of this result. Normal ordering involves placing annihilation operators to the right of creation operators, anti-normal ordering places them to the left, and symmetric ordering consists in a linear combination of both.

[23] The two-level atom is often used to discuss spontaneous emission and the Lamb shift, hence its importance.

[24] E^- is the negative frequency component of the field which involves the creation operator alone, whereas the positive frequency component E^+ involves the annihilation operator. The same holds for P^- and P^+.

to Q_0 and Q_S being Hermitian. So imposing that Q_0 and Q_S be Hermitian is not enough to remove the ordering freedom and the underdetermination associated with it. In addition, one needs to demand that Q be written in a form that involves products of only E with P,[25] not products of E^+ with P^+ (or E^- P^-) as above. That is, operators for the system and for the fields must be Hermitian operators throughout the derivation. The ordering that corresponds to this requirement is again the symmetric ordering.
Hence the symmetric one is the only ordering where:
- the vacuum field contribution, and the source field contribution to the effect, are represented by Hermitian operators.
- operators for the system and for the fields are Hermitian operators throughout the derivation.
For these reasons, it has been argued that the underdetermination is only superficial. The reasoning is that symmetric ordering should be the preferred choice, because observables are represented by Hermitian operators, so preferring symmetric ordering corresponds to imposing that the operators have physical meaning throughout the derivation. And symmetric ordering involves the vacuum field as well as the source field — in fact, equal contributions from both ([8], [14]). However, as will be discussed below, this line of reasoning has not met with universal agreement.

3 The Fluctuation-Dissipation Theorem (FDT)

3.1 Description

Aside from operator Hermiticity, another issue has played an important role in the way physicists have thought of the relative roles of the vacuum and source fields: the Fluctuation-Dissipation Theorem (FDT). The relation between the two fields has been viewed as a special instance of this theorem ([15]).
The FDT was proposed in 1951 by Herbert B. Callen and Theodore A. Welton. It generalizes a result derived by H. Nyquist in 1928, which relates voltage fluctuations to resistance in electric circuits ([16]). What fascinated Callen and Welton about Nyquist's relation and prompted them to generalize it was that it doesn't simply relate physical quantities for a given physical situation, but two different processes: it "relates a property of a system in *equilibrium* (i.e., the voltage fluctuations) with a parameter which characterizes an irreversible process (i.e. the electrical resistance)" ([15], p.34).
Callen and Welton considered an arbitrary dissipative system, i.e. a system able "to absorb energy when subjected to a time-periodic perturbation" ([15], p.34). Their FDT then states that, for such a system, the fluctuating force on it, $\mathscr{F}_f$ obeys:

$$\langle \mathscr{F}_f^2 \rangle = \frac{2}{\pi} \int_0^\infty R(\omega) E(\omega, T) d\omega, \tag{14}$$

[25] or products of appropriate combinations of $E^+ - E^-$ and $P^+ - P^-$. [8], p.1625.

where:
$$E(\omega, T) = \frac{1}{2}\hbar\omega + \hbar\omega[e^{(\frac{\hbar\omega}{kT})} - 1]^{-1}, \tag{15}$$
is the energy of the system, which corresponds to the expression for the mean energy of an oscillator of natural frequency ω, at temperature T, and the "generalized resistance" $R(\omega)$ characterizes the *dissipation*, given by ([15], p.37):
$$R(\omega) = Re\left(\frac{\mathscr{F}_d}{\dot{Q}}\right), \tag{16}$$
where $\dot{Q}$ is the response of the system, and $\mathscr{F}_d$ the dissipative force.[26]
Note the two terms in Eq.(15): the first term corresponds to zero-point energies, and in the $T = 0$ case $E(\omega, T)$ reduces to this term alone. In contrast, in the high temperature limit, $E(\omega, T)$ can be approximated by the second term which is then equal to kT.[27]
Applying the theorem requires identifying the generalized resistance, hence usually the dissipative force and the response in Eq.(16).

3.2 Examples

3.2.1 Nyquist's relation

By deriving the relation that led to Callen and Welton's generalization in the form of the FDT, Nyquist meant to account for the observation of an electromotive force ($e.m.f.$) in conductors in the absence of an external potential, due to their thermal agitation alone.
When we identify:
- the fluctuating force $\mathscr{F}_f$ with the electromotive force ($e.m.f.$)
- the generalized resistance $R(\omega)$ with the usual electric resistance

and we consider the high temperature limit where $E(\omega, T) \backsim kT$, the FDT (i.e. eq.14) becomes ([15], p.37):
$$\langle e.m.f.^2\rangle \simeq \frac{2}{\pi}kT\int_0^\infty R(\omega)d\omega, \tag{19}$$
in agreement with Nyquist's result.

[26] [15], p.35. More precisely, they considered a periodic perturbing force of the form:
$$\mathscr{F}_f = \mathscr{F}_{f0}\ sin(\omega t). \tag{17}$$
They found the power dissipated by the system to be proportional to the square of this perturbation, and related the proportionality constant to generalized notions of impedance $Z(\omega)$ and resistance $R(\omega)$:
$$Power = \mathscr{F}_f^2\frac{R}{Z}. \tag{18}$$
Callen and Welton actually did not distinguish between $\mathscr{F}_f$ and $\mathscr{F}_d$ as I have done here: they used the same symbol for both, suggesting that the force which fluctuates is the same force as that responsible for the dissipation. However in the examples that they discussed as applications of their theorem, the distinction is intuitively helpful, as we shall see below.

[27] This can be seen by using the binomial expansion on the second term with $kT << \hbar\omega$, which serves as the condition for the high temperature limit.

3.2.2 Brownian motion

Applying the FDT to Brownian motion (the random movements that small particles in a fluid exhibit) one recovers the known form of the fluctuating force on the particles. In this case the following identifications need to be made:
- $\mathscr{F}_f$: force(s) due to the fluid as its molecules hit the system,
- $\mathscr{F}_d$: viscous force, responsible for dissipation, and equal to $-\eta v$, where v is the speed of the particle and $\eta = 6\pi$ viscosity $\times$ particle radius,
- response $\dot{Q}$: speed v of the particles,
consequently, from Eq.(16):
- the generalized resistance $R(\omega)$ is equal to $-\eta$.
Then in the high temperature limit $E(\omega, T) \backsim kT$, the FDT (i.e. Eq.(14)) becomes:[28]

$$\langle \mathscr{F}_f^2 \rangle = \frac{2}{\pi} kT\eta \int d\omega, \tag{20}$$

in agreement with the known force responsible for Brownian motion.

3.2.3 Radiating charge

The two examples given above provide an intuitive feel for the FDT, notably because they are classical, as we took the high temperature limit in both cases. Of greater interest to our present concerns regarding electromagnetic fields however, Callen and Welton also applied their theorem to the radiation emitted by an accelerating charge. More specifically, they discussed a physical dipole oscillator ([15], pp.37-38).
They sought to find the force responsible for the fluctuations that their theorem implies must exist, knowing that a dissipative effect occurs due to the radiation reaction force, $-\frac{2e^2}{3c^3}\dddot{x}$, responsible for the energy loss of the charge. In order to do so, they again needed to find what the generalized resistance $R(\omega)$ is in the situation at hand.
In this case the relevant identifications are:
- $\mathscr{F}_f$: the fluctuating force on the charge,
- $\mathscr{F}_d$: the radiation reaction force, $-\frac{2e^2}{3c^3}\dddot{x}$, where $x = \frac{P_0}{e} sin(\omega t)$ with P_0 the dipole moment of the oscillator at maximum amplitude,
- $\dot{Q}$: the speed of the charge $\dot{x}$,
consequently from Eq.(16):
- the generalized resistance $R(\omega)$ is equal to $-\frac{2}{3}\frac{e^2\omega^2}{c^3}$.

What the FDT implies is that there must be on the charge a randomly fluctuating force $\mathscr{F}_f = e\mathscr{E}_x$, of the form:

$$\langle \mathscr{F}_f^2 \rangle = \langle e^2 \mathscr{E}_x^2 \rangle = \frac{2}{\pi} \int_0^\infty E(\omega, T) \; \frac{2}{3} \frac{e^2\omega^2}{c^3} \; d\omega \tag{21}$$

This time Callen and Welton did not take the high-temperature limit to recover a classical result, but kept the exact form of $E(\omega, T)$ (Eq.(15)), including the

[28] $\langle \mathscr{F}_f^2 \rangle = \frac{2}{\pi} \int_0^\infty R(\omega) E(\omega, T) d\omega$

zero-point energy term $\frac{1}{2}\hbar\omega$ that it would reduce to at $T = 0$. They noted that the energy density of an isotropic radiation field is directly related to the fluctuating electric field by:

$$\text{energy density} = \frac{\langle \mathscr{E}^2 \rangle}{4\pi} = \frac{3}{4\pi}\langle \mathscr{E}_x^2 \rangle \tag{22}$$

and therefore the FDT directly leads to:

$$\text{energy density} = \frac{3}{4\pi e^2}\langle \mathscr{F}_f^2 \rangle = \frac{1}{\pi^2 c^3}\int_0^\infty \left(\frac{1}{2}\hbar\omega + \hbar\omega\left[exp\left(\frac{\hbar\omega}{kT}\right) - 1\right]^{-1}\right)\omega^2 d\omega. \tag{23}$$

This energy density clearly involves two contributions:
- the Planck radiation law, given by the second term:

$$\frac{1}{\pi^2 c^3}\int_0^\infty \hbar\omega^3[exp(\frac{\hbar\omega}{kT}) - 1]^{-1} d\omega, \tag{24}$$

- the zero-point energy term, to which the energy density reduces in the limit of $T \to 0$ (but which is also present at non-zero temperatures):

$$\int_0^\infty \frac{\hbar\omega^3}{2\pi^2 c^3} d\omega. \tag{25}$$

Recall that the spectral energy density of the vacuum $\rho_0(\omega)$ has the form:

$$\rho_0 = \frac{\hbar\omega^3}{2\pi^2 c^3}. \tag{26}$$

So the FDT states that the fluctuating force due to the zero-point energy term involves an expression of the same functional form as the spectral energy density of the vacuum $\rho_0(\omega)$.

3.3 Possible interpretations

Looking at Eqs.(23) and (25), one would think that the implications of the FDT for the existence of vacuum fluctuations of the electromagnetic field are quite straight forward. Fluctuations in this field occur as thermal effects at non-zero temperature, but do not vanish at $T = 0$ when the field is in its ground state.
Of course in view of its derivation, using the result of the FDT for a radiating charge as evidence for the existence of vacuum fluctuations certainly does not seem free of circularity: we have used a zero-point contribution in Eq.(15) for $E(\omega, T)$, so we should hardly be surprised that we get one in our final result. We would get one too for the Nyquist circuit and Brownian motion if we did not choose to neglect it by taking the high temperature limit.
Yet when we consider the physics literature pertaining to debates on the issue, we do find disagreement regarding the implications of the FDT for the existence of vacuum fluctuations. Associated to those, we also find differences in the way the FDT has been interpreted. It is not even always clear from the outset

whether the FDT (whatever its interpretation) is acting as a premise in the discussions. Often, it seems more like the considerations in the light of which it is being interpreted are themselves put forward as arguments for or against the idea of vacuum fluctuations. It is these interpretations that we now turn to.
The classical, high temperature limit applications of the FDT discussed above provide us with an intuitive grasp on the meaning of this theorem. Peter Milonni has described the latter as follows:

> If a system is coupled to a "bath", that can take energy from the system in an effectively irreversible way, then the bath must also cause fluctuations. The fluctuations and the dissipation go hand in hand, we cannot have one without the other. ([12], p.54)

In the case of the electric circuit discussed by Nyquist, the dissipative force and the fluctuating force are both exerted on the electrons by the ionic lattice of the metal, so that the system is best described to be the electrons, and the bath the lattice.[29]
For Brownian motion, the dissipative effect is due to the viscous force exerted on the particle by the fluid, and the fluctuating forces are also exerted by the molecules of the fluid as they hit the particle. So in this case the system is the particle, and the bath is the molecules of the surrounding fluid.
This is summarized in Table 1, and we note that in these cases the fluctuating and dissipative forces are both ultimately exerted by the same entity (i.e. metal lattice, fluid molecules).[30]
The question of interest to us is: if we now try to give an analogous description of the FDT for a radiating charge, what can it teach us, if anything, about the radiation reaction field, the vacuum field fluctuations, and the relationship between them?
Here the dissipative effect is due to radiation reaction, so the dissipative force is exerted by the radiation reaction field. This is really only a matter of definition, since the radiation reaction field is a concept that was introduced in order to account for the (average) energy loss as a charge radiates.
Since this time we did not take the high-temperature limit, we get two contributions to the fluctuating forces: the electromagnetic force due to the Planck radiation field, and the electromagnetic force $\mathscr{F}_{f0}$. Whether the latter can be

[29]In fact Nyquist himself considered two conductors. In his scheme, the fluctuations originate in one conductor and dissipation in the other: the dissipative effect is the heat generated in the latter, which Nyquist identified as due to *e.m.f.* fluctuations in the circuit that are ultimately ascribed to thermal agitation in the first conductor. But naturally the distinction between these two conductors is artificial, meant to associate different parts of the one and same system, i.e. the circuit, with the two distinct processes of fluctuation and dissipation, for the sake of conceptual clarity when discussing these processes. In reality both fluctuation and dissipation occur throughout the resistive circuit.

[30] This common/ source of the forces is presumably the reason why Callen and Welton did not distinguish between $\mathscr{F}_f$ and $\mathscr{F}_d$ in their derivation, representing both by $\mathscr{F}$. The rationale for distinguishing between $\mathscr{F}_f$ and $\mathscr{F}_d$ as I have done is that they play different roles, occurring in the context of qualitatively different processes: $\mathscr{F}_d$, being dissipative, directly increases entropy, whereas $\mathscr{F}_f$ does not — it pertains to an equilibrium situation in so far that its expectation value is zero.

	System	Bath	Generalized Resistance	Dissipative Force $\mathscr{F}_d$	Fluctuating Force $\mathscr{F}_f$
Circuit discussed by Nyquist	electrons	ionic lattice of the metal	electrical resistance	• origin: due to thermal agitation • exerted by: the metal lattice	• origin: e.m.f. fluctuations due to thermal agitation • exerted by: the metal lattice
Brownian Motion	particle	molecules of the fluid	viscosity	• origin: viscous force due to thermal agitation • exerted by: the fluid	• origin: impacts from the surrounding molecules • exerted by: the fluid

Table 1: The FDT applied to the Nyquist circuit and Brownian motion

ascribed to the vacuum electromagnetic field is what is at stake.
Now for the non-zero temperature contribution, the dissipative force and the fluctuating force are exerted by the same entity: the field that exerts a reaction on the charge as the charge emits is the field it emits (see Table 2). Hence the bath is the field radiated by the charge, the source field.
For the $T = 0$ contribution though, things get more interesting. The relevant dissipative force is the same as for the non-zero temperature contribution, since in deriving their result Callen and Welton substituted for it only the radiation reaction force and no other contributions. So it is again the field radiated by the charge. What does this imply for $\mathscr{F}_{f0}$?
If we take the FDT to entail that the fluctuating force is exerted by the same entity as the dissipative force, then we are led to the conclusion that $\mathscr{F}_{f0}$ is due to the radiation reaction field, i.e. to the radiated, source field. This could be argued to be the natural interpretation of the FDT, in so far that looking at Eqs.(23), (25) and at our table, it seems clear that both thermal and zero-point effects have the same relation to the dissipative force, hence to the radiation reaction field. So if the fluctuating force associated with thermal effects is due to the radiated field, why not that associated with zero-point effects? Furthermore, one usually thinks of the FDT as involving a system and one, single bath that exerts both $\mathscr{F}_d$ and $\mathscr{F}_f$.
Then again, most applications of the FDT involve classical physics, hence the high temperature limit. So perhaps one should not take for granted that the identity of $\mathscr{F}_d$ and $\mathscr{F}_f$ necessarily holds for the zero-point energy contributions. And if one then takes the stance that the FDT does not require both types of

	System	Bath	Generalized Resistance	Dissipative Force $\mathscr{F}_d$	Fluctuating Force $\mathscr{F}_f$
Radiating charge	the charge	?	$-\frac{2}{3}\frac{e^2\omega^2}{c^3}$	• origin: radiation reaction • exerted by: the radiation reaction field $-\frac{2e^2}{3c^3}\dddot{x}$ i.e. the electromagnetic field radiated by the charge	• origin: - non-0 temperature contribution: Planck radiation - $T = 0$ contribution: zero-point effects • exerted by: - non-0 temperature contribution: the Planck electromagnetic field i.e. the electromagnetic field radiated by the charge - $T = 0$ contribution $\mathscr{F}_{f0}$: the zero-point electromagnetic field i.e. the electromagnetic field in its ground, "vacuum" state ?? Or radiation reaction field ?? In either case: $\int_0^\infty \frac{\hbar\omega^3}{2\pi^2c^3}d\omega$

Table 2: The FDT applied to a radiating charge

forces to be exerted by the same entity, then one is tempted to identify $\mathscr{F}_{f0}$ as due to the vacuum field.

It is interesting to compare these two alternatives to the high-temperature examples we discussed, since being classical they are intuitive. There is an interesting difference between the bath for the Nyquist circuit and Brownian motion on the one hand, and for the Planck radiation on the other. In the former two cases, the bath is a medium that exists independently of the system — and as a result the dissipative forces exerted on the system are clearly external forces, whose appearance the system had nothing to do with. By contrast, in the case of Planck radiation, the bath has been emitted by the system itself — hence the dissipative force being a reaction force. Attributing $\mathscr{F}_{f0}$ to the vacuum field or to radiation reaction entails a similar difference, which is arguably the main, defining difference between the two fields: the vacuum field is thought to exist irrespective of whether any charge is or has ever been accelerating in it.

3.4 Interpretations of the Fluctuation Dissipation Theorem in the Physics literature

When we consider the physics literature on the topic, disagreements center around whether $\mathscr{F}_f$ and $\mathscr{F}_d$ can be said to be exerted by the same entity, for the $T = 0$ contribution, and whether or not the fluctuating forces $\mathscr{F}_f$ are due to vacuum field fluctuations.[31]

Perhaps surprisingly, the discussion has not primarily focused on whether the FDT necessarily implies that $\mathscr{F}_f$ and $\mathscr{F}_d$ are both exerted by the same entity, with the view to then draw conclusions regarding the specific case of an accelerating charge. In fact, it is often unclear whether the FDT is actually playing the part of premise in the discussions, and it seems interpreted in light of other considerations, themselves put forward as arguments for or against the idea that the vacuum field is responsible for $\mathscr{F}_f$. We now examine these arguments, and what criteria have been implicitly used to assess the origin of $\mathscr{F}_f$.

3.4.1 The FDT implies the existence of vacuum fluctuations

3.4.1.1 Callen and Welton Most physicists who have worked on the issue seem to have held the fluctuation-dissipation theorem to imply the existence of the vacuum field, understood as a distinct entity from the radiation reaction one. This certainly appears to have been true of Callen and Welton themselves. Indeed, after deriving Eq.(23), they commented on their result:

> The first term in this equation gives the familiar infinite "zero-point" contribution, and the second term gives the Planck radiation law ([15], p.38).

Hence they identified $\mathscr{F}_f$ as being due in part to the vacuum field, and the criterion they used as the basis for this identification was the functional expression

[31] More precisely, whether they are partly due to vacuum fluctuations in the general case, and entirely so at $T = 0$.

of their spectral energy density: the expression for the spectral energy density of the fluctuating field predicted by the FDT (i.e. the integrand of Eq.(25)) is identical to the known form of the spectral energy density of the vacuum field.

3.4.1.2 Peter Milonni Peter Milonni is a physicist who has devoted a lot of his research to the role of radiation reaction and the vacuum field in the various physical effects thought to involve them, and notably to the issue of operator ordering discussed in the previous section. He takes pains to make his interpretation of the FDT explicit. The full text of the excerpt quoted earlier reads:

> What we have here[32] is an example of a "fluctuation-dissipation relation." Generally speaking if a system is coupled to a "bath" that can take energy from the system in an effectively irreversible way, then the bath must also cause fluctuations. The fluctuations and the dissipation go hand in hand; we cannot have one without the other. In the present example the coupling of a dipole oscillator to the electromagnetic field has a dissipative component, in the form of radiation reaction, and *a fluctuation component, in the form of the zero-point (vacuum) field*; given the existence of radiation reaction, the vacuum field must also exist in order to preserve the canonical commutation rule and all it entails.[33]

So Milonni takes the FDT to imply the existence of the vacuum field, as a distinct physical entity from the radiation reaction field, and attributes $\mathscr{F}_f$ to the vacuum field.
Yet he does not base these opinions on the FDT alone. His mention of "the canonical commutation rule" refers to a derivation that precedes the remarks just quoted. We shall discuss it in detail in section 4, but it should already be said that it does not make use of the FDT. Furthermore, in addition to this derivation involving the commutation rule, Milonni must also have in mind the issue of operator ordering, which has been the focus of much of his research. Unlike the FDT *per se*, this is an approach where the radiation reaction field and the vacuum field can easily manifest themselves as compatible alternatives as much as mutually exclusive ones, since the ordering can lead to contributions from both fields as readily as from one of them.

3.4.1.3 Dennis Sciama In the chapter that he contributed to *The philosophy of vacuum* ([14]), Sciama discussed a number of effects involving "zero-point energy" and "zero-point noise", including of the electro-magnetic field. Of greatest interest here is his treatment of the transition of a two-level atom from the excited to the ground state, accompanied by the emission of a photon.[34] As

[32]These remarks come after a discussion of the commutation relation for a dipole oscillator.
[33][12], p.54. My emphasis.
[34][14], pp.148-150. This is what is usually called spontaneous emission, but Sciama prefers avoiding the phrase because whether it is appropriate is precisely what is at stake.

in Callen and Welton's work, we are dealing with a radiating charge; only this time it is clearly a bound one, i.e. an electron in an atom.

Sciama holds that it is in this specific context that disagreement regarding the vacuum field has occurred among physicists.[35] He phrases the issue in a slightly different way: instead of asking what *field* (radiation or vacuum field) is responsible for the transition through playing the part of $\mathscr{F}_{f0}$, he asks what *process* is: stimulated emission or spontaneous emission. In fact the two approaches are similar: "stimulated emission" is "a reaction to spontaneous radiation emitted by the atom" whereas so-called "spontaneous emission" is thought stimulated by the vacuum field.
Sciama argues that the relative contributions of stimulated and spontaneous emission to the transition rate are "precisely equal" and "each contribution is physically real".
He further argues that the processes occur even for an atom in its ground state, In this case he speaks of emission vs *absorption* rather than in terms of stimulated vs *spontaneous* emission.[36] The way he visualizes the situation in this context suggests he has the FDT or related considerations in mind: he speaks of the atom "radiating to and absorbing energy from the zero-point fluctuations of the vacuum electromagnetic field", and further states: the radiation rate is determined by the noise power in the zero-point fluctuations of the atomic dipole moment, and the absorption rate is determined by the noise power in the zero-point fluctuations of the electromagnetic field ([14], p.149).[37]
What interests us are his reasons for thinking that both stimulated and spontaneous processes (hence both the vacuum and the radiation reaction fields) make "precisely equal" contributions to the effect. Although he thinks of the physical situation in terms of a system in a bath, and certainly has the FDT in mind, his argument does not *rely* on the FDT but on considerations related to operator ordering, i.e. that only symmetric ordering involves only Hermitian operators.([14], p.148).

3.4.1.4 J. Dalibard J. Dupont-Roc and C. Cohen-Tannoudji Much of Sciama's conclusions are motivated by the work of J. Dalibard, J. Dupont-Roc and C. Cohen-Tannoudji, in which the latter argue that symmetric ordering is more physically correct than other orderings on the basis of the Hermitian character it implies for the operators. Besides this consideration however, they have other reasons for preferring symmetric ordering, which are related to the FDT. Indeed they argue that when choosing symmetric ordering, one finds physically intuitive results regarding properties of the system S and reservoir R

[35]This is likely due to the fact that this was the effect specifically treated by Jaynes to criticise the notion that electromagnetic vacuum fluctuations are real, as we shall discuss shortly.

[36]The switch in language naturally corresponds to the fact that what the processes now need to account for is no longer an emission, hence a net loss of energy by the atom, but the lack thereof, hence a dynamic equilibrium of loss and gain.

[37]Sciama's phrasing suggests that he thinks of the bath to consist of the vacuum field alone, but this is doubtless because he is now talking about an atom in its ground state, which does not emit radiation.

— i.e. the bath.[38]
Their treatment is meant to apply even more generally than the FDT in so far that it is valid for "an arbitrary state of the reservoir", rather than requiring that it be at thermal equilibrium ([8], p.13). At the same time, the discussion is given with the Lamb shift in mind. They derive expressions for the energy shift of the system due to the reservoir, $(\delta E_a)_{rf}$, and due to self-reaction, $(\delta E_a)_{sr}$.[39]
They find, respectively:

$$(\delta E_a)_{rf} = -\frac{1}{2}\sum_{ij}\int_{-\infty}^{+\infty} d\tau \;\; C_{ij}^{(R)}(\tau)\; \chi_{ij}^{(S)}(\tau) \tag{27}$$

$$(\delta E_a)_{sr} = -\frac{1}{2}\sum_{ij}\int_{-\infty}^{+\infty} d\tau \;\; \chi_{ij}^{(R)}(\tau)\; C_{ij}^{(S)}(\tau), \tag{28}$$

where:
- the C_{ij} refer to correlation functions, that describe the dynamics of fluctuations in either the reservoir $C_{ij}^{(R)}$ or the system $C_{ij}^{(S)}$
- the χ_{ij} refer to linear susceptibilities, and represent the response of the system $\chi_{ij}^{(S)}$ or reservoir $\chi_{ij}^{(R)}$ to a perturbation.
They stress that these results call for a simple physical interpretation:

> One can consider that the fluctuations of R [the reservoir], characterized by $C_{ij}^{(R)}(\tau)$, polarize S [the system] which responds to this perturbation in a way characterized by $\chi_{ij}^{(S)}(\tau)$. The interaction of the fluctuations of R with the polarization to which they give rise in S has a non zero value because of the correlations which exist between the fluctuations of R and the induced polarization in S. [...] The same comments can be made [for Eq.(28)] as for [Eq.(27)], the roles of [the system] and [reservoir] being interchanged. ([8], p.12.)

They summarize this physical model by the diagram:

[38] [8], p.1626. They ask: "Is it possible to understand the evolution of S [the system] as being due to the effect of the reservoir fluctuations acting upon S, or should we invoke a kind of self-reaction, S perturbing R which reacts back on S?" and note that "For spontaneous emission, [...] the vacuum field, with its infinite number of modes, plays the role of R."

[39] More generally, they consider the average rate of the system variable G, $\left\langle \frac{dG(t)}{dt} \right\rangle_R$ where the subscript R indicates that the average is taken over the states of the reservoir. They identify the contributions that the vacuum field and self reaction make to this rate — respectively represented by $\left\langle \left(\frac{dG(t)}{dt}\right)_{vf} \right\rangle_R$ and $\left\langle \left(\frac{dG(t)}{dt}\right)_{sr} \right\rangle_R$.[40]

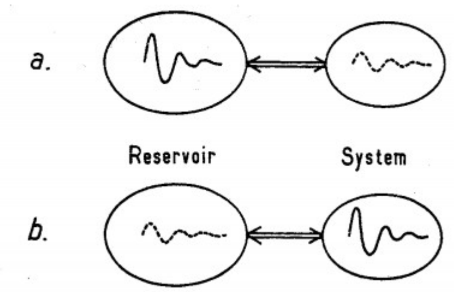

Fig. 2. — Physical pictures for the effect of reservoir fluctuations and self reaction. *a)* Reservoir fluctuations : the reservoir fluctuates and interacts with the polarization induced in the small system. *b)* Self reaction : the small system fluctuates and polarizes the reservoir which reacts back on the small system.

Fig. 1: Interaction between system and reservoir ([8], p.12.)

3.4.2 The FDT does not imply the existence of vacuum fluctuations

3.4.2.1 Edwin Jaynes Edwin T. Jaynes argued that effects such as the Lamb shift and spontaneous emission could be derived within what he termed "neoclassical theory", which meant that matter was indeed quantized but the electromagnetic field was not, in obvious contrast with QED.[41] He did interpret the radiation reaction field as an operator, but as "an operator not on the "Maxwell Hilbert space" of a quantized field, but on the "Dirac Hilbert space" of the electrons", and he saw no necessity for the "Maxwell Hilbert space" to account for the effects of interest ([17], pp.10, 18).
Unlike the physicists we just discussed, Jaynes insisted that the FDT does not imply the existence of vacuum fluctuations ([17]). In a provocative talk given at the Conference on Coherence and Quantum Optics in 1977, he stated:

> This independence of the initial ordering is, then, just a very simple, general, and elegant fluctuation-dissipation theorem; but let me suggest a different physical interpretation from the usual one. This complete interchangeability of source-field effects and vacuum-fluctuation effects does not show that vacuum fluctuations are "real". It shows that source field effects are the same as if vacuum fluctuations were present.
> For many years, starting with Einstein's relation between diffusion coefficient and mobility, theoreticians have been discovering a steady stream of close mathematical connections between stochastic problems and dynamical problems. It has taken us a long time to recog-

[41] This made it a semi-classical theory. What is particularly interesting about Jaynes' neoclassical theory is what motivated it. Peter Milonni has described it as "based on the recognition that rather few phenomena in non-relativistic radiation theory actually require field quantization for their explanation, and its purpose was to explore just how far one could get without field quantization and possibly to point the way to alternatives to QED." [12], p.148.

> nize that QED was just another example of this.
> But in another sense, I do have to concede that vacuum fluctuations are, after all, "very real things" ([17], p.12).

Jaynes only clarified these contradictory statements by discussing an example — precisely, the case of a spontaneously radiating atom. From it he eventually concluded:

> The radiating atom is indeed interacting with an EM field of the intensity predicted by the zero-point energy; but this is just the atom's own radiation reaction field ([17], p.13.).

Jaynes essentially argued that the presence in Callen and Welton's result of the expression $\frac{\hbar\omega^3}{2\pi^2c^3}$ should not be interpreted as evidence for the existence of vacuum fluctuations (Eq.(25)). Central to his argument is the idea that what is physically significant is not the *spectral* energy density $\rho(\omega)$, but $\rho(\omega)$ integrated over the appropriate frequencies, i.e. the energy density, W_f. Note that in Callen and Welton's result, $\rho(\omega)$ appears in an integral over all frequencies in the expression for the energy density, which comes directly from $\mathscr{F}_f$. Jaynes' reasoning consists in saying that not all of these frequencies are physically present. He instead considered $\rho(\omega)$ integrated only over the small frequency bandwidth $\Delta\omega$, which he deemed "effective in causing the atom to radiate" ([17], p.12).
Jaynes' argument involves deriving and comparing two expressions:
- W_{0eff}, the "effective" part of the energy density whose spectral energy is $\rho_o(\omega)$.
- W_{RReff}, the energy density of the radiation reaction field.
He found them to be given by the same expression:

$$\frac{1}{18\pi}\mu^2\left(\frac{\omega}{c}\right)^6, \tag{29}$$

where μ stands for the electric dipole moment of the relevant atomic transition, and ω for the natural line frequency of this transition.[42]
Jaynes' derivation shows that if one takes into account only some field modes and components (deemed physically significant), the energy density due to "$\rho_o(\omega)$", W_{0eff}, is given by the same expression as the energy density of the radiation field at the position of the atom, W_{RR}. From this he concluded that what had been interpreted as a contribution of the vacuum field is in fact due to radiation reaction.[43]
At this juncture it may be useful to ask what is meant by vacuum field in the context of radiating atoms and accelerating charges. Recall that strictly speaking, the vacuum field is the electromagnetic field in its ground state, free

[42]Jaynes' notation is not entirely consistent, as he first uses ω_0 to represent the natural line frequency, and simply ω in his final expressions for the two energy densities. However his derivation and reasoning require that he means the natural line frequency by this.

[43]This is most certainly the "other sense" in which he meant that vacuum fluctuations are "very real things", i.e. they are in fact the radiation reaction field.

of (real) particles. But in fact, as soon as a charge accelerates, it is bathed in its own radiation field, the real photons it is emitting. Phrases such as "vacuum fluctuations" and "zero-point (vacuum) field" can no longer refer *per se* to a vacuum surrounding the particle in this context, since the electromagnetic quantum field is no longer in its vacuum state. So how can we make sense of the phrase "vacuum field" in this context?

It seems fair to interpret these expressions as short-hand for "fluctuations above and below the (no-longer-zero-point) energy of the electromagnetic field". What can justify this use is that these fluctuations can be thought of as "the same ones" as would be there even if there were no real photons around, in the sense that they too result from the UR applied to the field.[44] So when one speaks of the contribution of the "vacuum field" in such a situation, one is referring to the contributions to the electromagnetic field that are due to the UR. And indeed, this is what makes the whole topic so fascinating: it involves possible implications of the UR. Being the result of the UR, vacuum fluctuations would be there whether or not there is radiation present, i.e. whether or not a charge has ever been accelerating. Essentially, the question Jaynes could be asking is: if we first imagine the universe completely empty of real particles (especially photons and accelerating charges), and we put in it an atom in an excited state, is this atom going to emit a photon? Would "spontaneous emission" take place, even then? This is what he means when he asks whether it is the vacuum field that is responsible for the effect.[45]

Quite obviously Jaynes believes that the answer is "no, an excited atom placed in a universe free of real photons would not emit a photon". It seems clear that his reason for thinking so is that, since the modes responsible for the emission are of the same frequencies as the radiation emitted when the atom does, it is a more economical hypothesis to suppose that these modes in fact belong to the field we know for a fact exists anyway. However for Jaynes this economy goes beyond a concern with Occam's razor. It does away with a crucial difficulty, i.e. the infinities known to plague QED; no vacuum field, no infinities:

> The fantastic numbers noted before disappear as soon as we realize that, in order to account for spontaneous emission, there is no need for this energy density to be present in all space, at all times, in all frequency bands. It is produced automatically by the radiating atom, but in a more economical way; only the field component that is needed, where it is needed, when it is needed, and in the frequency band needed ([17], p.14).[46]

[44] More precisely, they result from commutation relations between field operators.

[45] This also explains why the issue of what field is responsible has been described in terms of whether the emission process is spontaneous or stimulated. If the modes involved are only those radiated by the atom, they are only a consequence of the process, not a cause of it, and it is truly spontaneous. By contrast if emission is caused by the vacuum field it is of course stimulated by the latter.

[46] An interesting aspect of Jaynes' demonstration is worth noting: $\hbar$ cancels out in his derivation of W_{0eff} — as evidenced by the final result. This occurs because his expression for the bandwidth $\Delta\omega$ is proportional to $\hbar^{-1}$. This in turn comes about because he finds $\Delta\omega$

Hence Jaynes interpreted the FDT from the physicists discussed earlier. He thought that the bath consists of only one field, the radiation field, alone responsible for $\mathscr{F}_f$. His view was not motivated by an independent understanding of the FDT, but by the other considerations just discussed, which then led to his interpretation of the FDT.

4 Consistency of the quantum theory of radiation: necessity of the vacuum field for the commutation relations to hold.

In addition to arguments in favor of a preferred ordering, and considerations relating to the FDT, other evidence pertaining to the vacuum field involves the position-momentum commutation relation. Indeed a very interesting demonstration consists in showing that for $[\mathbf{X}, \mathbf{P}] = i\hbar$ to hold for an accelerating charge — which of course undergoes radiation reaction damping in the process —, the charge needs to be driven by the vacuum field. And not only are both vacuum and radiation fields needed, they also need to be related to one another in a specific way.[47]

Peter Milonni showed this notably by considering a non-relativistic electron in free space. The Heisenberg equation of motion for the electron is:

$$\frac{d^2\mathbf{X}}{dt^2} - \frac{2e^2}{3mc^3}\frac{d^3\mathbf{X}}{dt^3} = \frac{e}{m}\mathbf{E}_0,^{48} \tag{30}$$

where we recognize the second term as the radiation reaction force.

If this equation is solved for $\mathbf{X}$, and we then use it to find as well $\mathbf{P} = m\dot{\mathbf{X}}$, it turns out that the commutator of these operators takes the form:

$$[\mathbf{X}, \mathbf{P}] = \frac{8\pi^2 i}{3m}\int_0^\infty d\omega \frac{\rho_0(\omega)}{[\omega^3(1+\gamma^2\omega^2)]}, \tag{31}$$

where $\gamma = \frac{2e^2}{3mc^3}$ and $\rho_0(\omega)$ is the spectral density of the vacuum field. When we then substitute $\rho_0(\omega) = \frac{\hbar\omega^3}{2\pi^2 c^3}$ in this expression, we find the commutator equal

to be proportional to the Einstein A-coefficient (i.e. the coefficient of spontaneous emission, A_{21} is the probability per unit time that an atom in energy state 2 will spontaneously emit a photon and undergo a transition to energy state 1) and the latter is proportional to $\hbar^{-1}$. Indeed, Jaynes reasons that although the spectral density $I(\omega)$ of the radiation spontaneously emitted by the atom has a Lorentzian profile, and therefore no well-defined width, $I(\omega_0)\Delta\omega$ must be equal to the total energy radiated by the atom, $\int I(\omega)d\omega$, and the latter naturally involves the A-coefficient. This line of reasoning assumes that the bandwidth $\Delta\omega$, which refers to the frequencies of the fluctuating field causing the atom to radiate, can be identified with the width of the natural emission line of the atom. Jaynes does not strongly defend this position, merely stating that this is "presumably" the case. Perhaps he felt that his final result, i.e. the identity of expressions for W_{0eff} and $\mathscr{W}_{RR}$, was remarkable enough to lend credence to the assumptions he had to make in order to reach it.

[47] [18], p.106; [19], p.1322, 1323; [20].

[48] Note that E_0 here is the homogeneous solution of the Maxwell (Heisenberg) equation for the electric field.

to $i\hbar$ as it should be.
The crucial point is this: had we neglected the vacuum field in Eq.(30) by setting the RHS to zero, $[\mathbf{X}, \mathbf{P}]$ would not be equal to $i\hbar$. Nor would it be if the field responsible for the driving force had a spectral energy density different from $\frac{\hbar\omega^3}{2\pi^2 c^3}$. Notably, the energy spectrum needs to go as the *third* power of the frequency ω because the radiation reaction force involves the *third* derivative of the position.[49]
One may perhaps wonder if there isn't some circularity at play in this argument. After all, the consistency with quantum mechanics at stake here is the commutation relation that formally expresses the UR, and it is again the UR that lead to the prediction of vacuum fluctuations in the first place. Why should we now regard another *formal* argument based on the *same* "principle",[53] as additional and (as Milonni implies) even stronger evidence for vacuum fluctuations? However the commutation relation at stake here is not the one that gives rise to the zero point energy of the field in QFT: it is the NRQM commutation relation for the charge. Now, one can argue that we are still using the same "principle", even though it now refers to different variables — position and momentum, vs field and canonical momentum. However as we shall see below there is an important difference between these two sets of variables, which may justify regarding the NRQM UR to be on a stronger footing than its QFT counterpart: quantum field and canonical momentum are not observable, in so far that they have zero expectation value.

5 Discussion

Do the arguments provided by Dalibard et al and Peter Milonni invalidate Jayne's view? There are at least two separate issues at stake: first, whether the ordering of operators can still be said to lead to underdetermination regard-

[49] The reverse is also true: had we instead neglected the radiation reaction effect in Eq.(30) by setting the second term to zero, $[\mathbf{X}, \mathbf{P}]$ would not be equal to $i\hbar$ either.
Milonni also gives elsewhere an analogous treatment for the case of a dipole oscillator.[50] There he uses the small-damping approximation $\dddot{\mathbf{x}} \cong -\omega_0^2 \mathbf{x}(t)$ in the equation for the Heisenberg-picture position operator $\mathbf{x}(t)$:

$$\ddot{\mathbf{x}} + \omega_0^2 \mathbf{x} - \tau \dddot{x} = \frac{e}{m}\mathbf{E}_0(t) \tag{32}$$

thereby obtaining:

$$\ddot{\mathbf{x}} + \tau\omega_0^2 \dot{\mathbf{x}} + \omega_0^2 \mathbf{x} \cong \frac{e}{m}\mathbf{E}_0(t).^{51} \tag{33}$$

As he notes:

> Without the free field $E_0(t)$ in this equation the *operator* $x(t)$ would be exponentially damped, and commutators like $[z(t), p_z(t)]$ would approach zero for $t >> (\tau\omega_o^2)^{-1}$. With the vacuum field included, however, the commutator is $i\hbar$ at all times.[52]

In other words, the presence of $E_0(t)$ is required for the variables that describe the dipole to take on their quantum character: without this field, their commutator is no longer proportional to $\hbar$; instead it vanishes so these variables would then commute as they do classically.
[53] In so far that the UR are often described as a principle.

ing which field is responsible for the effects of interest, and second, when we interpret an expression as being a signature of the vacuum field, are we sure that it is indeed the vacuum field which is at stake?

5.1 Operator Ordering

As explained above, the ambiguity regarding whether the vacuum field is definitely involved in the various relevant effects can be said to arise from issues regarding the commuting properties of operators, hence our freedom or lack thereof in ordering them. In so far that operators commute we get similar results in terms of numerical predictions, in so far that they don't we get different accounts of which field is responsible for the effects.[54]

This underdetermination does not allow for the *source field* to play no part in all effects, because in spontaneous emission all orderings involve at least some contribution from it. However, in and of itself, the ordering freedom does leave open the possibility that the *vacuum field* may play no part — hence that these effects provide no evidence for its existence.

As explained above, it has been argued that this underdetermination is only superficial, because symmetric ordering should be preferred to all other choices, and it involves equal contributions from the vacuum field as well as the source field. The proponents of symmetric ordering argue that it should be preferred because only with this ordering do we find that:

- the vacuum field contribution to the effect, and the source field contribution to it, are represented by Hermitian operators.
- throughout the derivation, operators for the system and for the fields are Hermitian operators (only products of Hermitian operators are involved, not products of, say, creation operators).

These two considerations are then elevated to the status of requirements to be imposed on derivations in order for them to make physical sense. The argument is that because in NRQM observables are represented by Hermitian operators, demanding that the operators be Hermitian throughout the calculation ensures that they have physical meaning throughout.

It should be noted that there may be some circularity in the reasoning involved, in so far that these two "requirements" do not appear as desiderata purely for their own sake, with lifting the underdetermination a mere fortunate consequence of them. In fact the latter, i.e. removing the ambiguity, also functions as an argument in favor of preferring one ordering over the others, and motivates seeking criteria to do so. Whether or not operators are Hermitian turns out to depend on the ordering, and has been deemed a good criterion to go by.

What can perhaps be argued to be a fortunate consequence of these requirements is the extent to which they lead to an intuitive, classical description of

[54] Recall that it is the operators for the system and the *total* field that commute, so whatever way we choose to order them, we get the same final, overall results, but that the operators for the vacuum and source field do not separately commute with operators for the system, so that different choices of ordering involve different contributions from the vacuum and source fields.

the relationship between system and vacuum field, whereby they can be said to polarize one another on the basis of Eq.(27) and Eq.(28).
However, not all researchers agree that these considerations justify regarding symmetric ordering as *correct*, or simply *more correct*, than the alternative choices. Ultimately, the difference between the different view points lies in the ontological status one chooses to ascribe to the positive (or negative) frequency part of a field. The physicists who argue for the superiority of symmetric orderings state:

> It would be difficult to elaborate a physical picture from an expression involving only the positive frequency part of the field which is not observable ([8], p.9).

Others, notably Peter Milonni, frame the issue in terms of the distinction between quantum and classical features, and refrain from ascribing greater import to the latter:

> Various [...] orderings give different weights to the vacuum and source fields when we try to interpret the results of a calculation. To emphasize as much as possible the classical-like aspects of the vacuum and source fields, we can choose a symmetric ordering at every stage of a calculation ([18], p.7).

Taking this view, one is then led to describe the phenomena of interest as due, say, to only the positive frequency part of the field when normal ordering is chosen, and this physical picture should be taken as seriously as the one described by normal ordering.[55]
Also, one can contrast Dalibard et al's concern with restricting the derivation to observables in the formal sense of the word, with Jaynes' operationalist motivations for doing away with the vacuum field, i.e. the at best indirect character of its experimental observability. Jaynes states:

> It seems to me that, if you say radiation is "real," you ought to mean by that, that it can be detected by a real detector. But an optical pyrometer sees only the Planck term, and not the zero-point term, in black body-radiation.[56]

[55] Normal ordering corresponds to:

$$E^{(-)}P + PE^{(+)}, \tag{34}$$

and $E^{(-)}$ acting on the left is equivalent to $E^{(+)}$ on the right.
It is worth noting that despite these differences of opinion, Peter Milonni actually does think that the vacuum field exists as well as the source field — simply he does not think so on the basis of the arguments that favor symmetric ordering.

[56][17], pp.5-6. For interest's sake, Jaynes goes on: "Of course, a staunch defender of present theory will say immediately that such objections reflect only naive metaphysical preconceptions of reality, not unlike pre-relativistic notions of absolute simultaneity, of just the kind that the Copenhagen interpretation of quantum theory has recognized. and rightly removed from science. ...
It is a supple ontology which supposes that vacuum fluctuations are just real enough to shift

One gets the impression that ultimately, for Dalibard et al and Sciama, it is the intuitive description of the relationship between system and vacuum field that convinces them of the correctness of symmetric ordering. It is arguably in this sense that the FDT can be said to provide a strong argument in favor of the existence of vacuum fluctuations, for it provides a classical framework in which Dalibard et al can easily interpret their results.

5.2 Are we really dealing with the vacuum field?

A key point of Jaynes' discussion was that one could easily be mistaken in identifying the vacuum field. Is this point possibly relevant to the arguments put forward by Dalibard et al and Milonni?

As shown above, Jaynes himself used the accepted expression for the spectral energy of the vacuum, "$\rho_o(\omega)$" i.e. $\frac{\hbar\omega^3}{2\pi^2c^3}$. His work stressed that the mere appearance of this expression in a result does not constitute evidence for the existence of the vacuum field. It seems fair to say that he agreed this expression represented the energy density that the vacuum field would indeed have per mode, if these modes did exist. He simply saw no reason in the appearance of this expression to assert that they do, in fact, exist. In his derivation, the piece of formalism relevant to their existence or lack thereof is the range of the frequencies one integrates $\rho_o(\omega)$ over. The vacuum field presumably involves an infinite range of frequencies, whereas the emitted radiation involves only the frequencies emitted by the atom. To account for observable facts, Jaynes argued, the appropriate range of frequencies is the latter. This did not show that the vacuum field could not be responsible, but it did show that its existence was not required.

Now Milonni's argument involving commutation relations certainly relies on identifying $\frac{\hbar\omega^3}{2\pi^2c^3}$ as the spectral density of the vacuum field. So does Jaynes point invalidate Milonni's argument?

The expression used by Milonni involves an integral over all frequencies, not simply some of them, and his result relies on performing this integral.[57] So his result does depend on the range of frequencies he considers. This suggests that Jaynes' point (i.e. that the field at stake may in fact not be the vacuum field) does not affect Milonni's result.

It does not seem to affect Dalibard et al.'s arguments either, although matters are less straightforward in this case: the spectral energy density $\frac{\hbar\omega^3}{2\pi^2c^3}$ does not play the role of an operational definition of the vacuum in their work. Instead, they define the vacuum field via the homogeneous solution to the field equations, i.e. the "free field". Now in classical electrodynamics such solutions exist and in no way correspond to what we mean by vacuum field: in that context, all fields have at some point in the past been emitted by some source. So considering

the hydrogen 2s level by 4 microvolts; but not real enough to be seen by our eyes, although in the optical band they correspond to a flux of over 100 kilowatts/cm^2. Nevertheless, the dark-adapted eye, looking for example at a faint star, can see real radiation of the order of 10^{-15} watts/cm^2."

[57]And not just on terms cancelling out in the integrand for instance.

the homogeneous solution alone is obviously no guarantee that we are dealing with the vacuum field. What makes it so is to add the condition that "no [real] photons are initially present" ([8], p.1618), which they impose through the requirement that "the radiation field is in the vacuum state at the initial time" ([8], p.1622). Hence it would appear that Jaynes' worry regarding the identification of the vacuum field does not apply to the work of Dalibard et al either.

5.3 Implication for the Uncertainty Relations

5.3.1 Preliminary: Interpretations of the Uncertainty Relations in the context of NRQM

The Uncertainty Relations have been interpreted in different ways. As Jan Hilgevoord and Jos Uffink note ([21], p.13):

> The interpretation of these relations has often been debated. Do Heisenberg's relations express restrictions on the experiments we can perform on quantum systems, and, therefore, restrictions on the information we can gather about such systems; or do they express restrictions on the meaning of the concepts we use to describe quantum systems? Or else, are they restrictions of an ontological nature, i.e., do they assert that a quantum system simply does not possess a definite value for its position and momentum at the same time? The difference between these interpretations is partly reflected in the various names by which the relations are known, e.g. as 'inaccuracy relations', or: 'uncertainty', 'indeterminacy' or 'unsharpness relations'.

Which interpretation of the UR one tends to adopt is generally related to which interpretation of quantum mechanics as a whole one favors. For this reason, Jan Hilgevoord and Jos Uffink deem as the "minimal interpretation" that which "seems to be shared by both the adherents of the Copenhagen interpretation and of other views".[58]
The minimal interpretation is a statistical one, with the uncertainties identified with spreads of probability distributions: one considers a very large number of copies of the system all formally described by the same state vector,[59] and look at the outcomes of, say, position (or momentum) measurements on all these copies. The UR are then taken to mean that "the position and momentum distributions cannot both be arbitrarily narrow" ([21], pp.34-35).
What makes it possible for this description to serve as a common-ground interpretation is the fact that it is purely epistemic in the sense that it only restricts what we can know about the systems, while remaining agnostic regarding the source of this restriction. In addition to forming such common-ground, it is

[58] [21], p.34. See also notably: [22], pp.82-83, [23].
[59] Which from an experimental standpoint implies that they must all be prepared in the same state.

also minimal in a second sense: it does "little more than filling in the empirical meaning" of an inequality which is widely considered as the formal expression of the UR:

$$\Delta_\Psi p \, \Delta_\Psi q \geq \frac{\hbar}{2} \tag{35}$$

where $\Delta_\Psi p$ and $\Delta_\Psi q$ are *standard deviations* of momentum and position, respectively:

$$(\Delta_\Psi p)^2 = \langle\Psi|p^2|\Psi\rangle - \langle\Psi|p|\Psi\rangle^2, \tag{36}$$

and similarly for position.[60]

This is a particular case of a more general theorem about Hermitian operators (A and B here):

$$\Delta_\Psi A \, \Delta_\Psi B \geq \frac{1}{2}|\langle\Psi| \, [A, B] \, |\Psi\rangle|.^{61} \tag{37}$$

Modern textbooks typically identify the UR with this inequality, which is the result of a formal derivation based on the quantum formalism itself, by contrast with Heisenberg's original formulation to which we shall soon turn. As can be seen from Eq.(37), the uncertainty involved is due to the commutation relations: when A and B commute, the RHS vanishes and the standard deviations can be simultaneously arbitrarily small; uncertainty only obtains when this is not the case.

The way Heisenberg originally discussed the UR had little to do with the spread of statistical distributions however.[62] Instead it was related to the inaccuracy of instruments. Both Heisenberg and Bohr saw in the UR more than a restriction on our ability to obtain information about quantum systems (of course in their view it entailed that too, but not only). This was the result of an operationalist standpoint: Heisenberg held that physical quantities (position, momentum) only have meaning if we can specify an experiment by which to measure them (what Hilgevoord and Uffink term the "measurement=meaning principle"). Restrictions in our ability to measure accurately (which *per se* is an epistemic issue) then imply corresponding inaccuracies in the *meaning* of physical quantities:

> If there are, as Heisenberg claims, no experiments that allow a simultaneous precise measurement of two conjugate quantities, then these quantities are also not simultaneously well-defined.[63]

[60] This was derived by Kennard in 1927, and Heisenberg presented it in his own work in 1930. [21], pp.19-20, [24], [25].

[61] This generalization was obtained by Robertson in 1929. [21], p.20.

[62] Even though he viewed in Kennard's result an exact proof of his UR, [21], p.20.

[63] [21], pp.6-8. In Heisenberg's view inaccuracies were related to the occurrence of discontinuous changes, i.e. in the thought experiment where he imagines trying to measure the position of an electron using light:

> At the instant of time when the position is determined, that is, at the instant when the photon is scattered by the electron, the electron undergoes a discontinuous change in momentum. This change is the greater the smaller the wavelength

But Heisenberg occasionally expressed views that could be described as more extreme, if denying something's *existence* can be deemed more extreme than denying the possibility of *defining* said thing. He famously wrote:

> I believe that one can formulate the emergence of the classical path of a particle pregnantly as follows: the path comes into being only because we observe it ([26], p.185),

on which Hilgevoord and Uffink comment:

> Apparently, in his view, a measurement does not only serve to give meaning to a quantity, it creates a particular value for this quantity. This may be called the measurement=creation principle. It is an ontological principle, for it states what is physically real.([21], p.12)

This in turn suggests an interpretation of the UR that can also be deemed ontological:

> Before the final measurement, the best we can attribute to the electron is some unsharp, or fuzzy momentum. These terms are meant here in an ontological sense, characterizing a real attribute of the electron.[64]

Bohr's understanding of the UR further differed from Heisenberg's. Notably, Bohr did not ascribe the UR's origin to discontinuous changes in momentum, but to his Principle of Complementarity:

> of the light employed, i.e., the more exact the determination of the position. At the instant at which the position of the electron is known, its momentum therefore can be known only up to magnitudes which correspond to that discontinuous change; thus, the more precisely the position is determined, the less precisely the momentum is known, and conversely.[26], pp.174-5, English translation [27], pp.62-84. See as well [21], p.7.

[64] [21], p.12. Hilgevoord and Uffink state this after summing up Heisenberg's views:

> First we measure the momentum of the electron very accurately. By measurement= meaning, this entails that the term "the momentum of the particle" is now well-defined. Moreover, by the measurement=creation principle, we may say that this momentum is physically real. Next, the position is measured with inaccuracy δq. At this instant, the position of the particle becomes well-defined and, again, one can regard this as a physically real attribute of the particle. However, the momentum has now changed by an amount that is unpredictable by an order of magnitude $|p_f - p_i| \sim \frac{\hbar}{\delta q}$. The meaning and validity of this claim can be verified by a subsequent momentum measurement.
> The question is then what status we shall assign to the momentum of the electron just before its final measurement. Is it real? According to Heisenberg it is not. Before the final measurement, the best we can attribute to the electron is some unsharp, or fuzzy momentum. These terms are meant here in an ontological sense, characterizing a real attribute of the electron.

> It is not so much the unknown disturbance which renders the momentum of the electron uncertain but rather the fact that the position and the momentum of the electron cannot be simultaneously defined in this experiment.[65]

Especially interesting is the derivation of the UR Bohr offered in 1928, in which uncertainties were defined in terms of the features of a wave packet.[66] Δx and Δt represented the spatial and temporal extensions of the wave packet, $\Delta\sigma$ and $\Delta\nu$ its range of inverse wave numbers ($\sigma = \frac{1}{k}$) and frequencies.
What is remarkable about this derivation is that it makes no use of commutation relations, in contrast to the now standard one discussed above. It relies instead on Fourier analysis to obtain the product of uncertainties, responsible for the restriction in defining quantities *simultaneously* (since as one factor increases the other must decrease) ([21], p.29):

$$\Delta x\,\Delta\sigma \approx \Delta t\,\Delta\nu \approx 1. \tag{38}$$

Then Bohr obtained the UR by substituting for frequency and wavenumber in these relations, using, respectively, the relation between energy and frequency $E = h\nu$, and between momentum and wavelength $p = \frac{h}{\lambda}$ ([21], p.29):

$$\Delta x\,\Delta p \approx \Delta t\,\Delta E \approx h. \tag{39}$$

Two remarks are in order regarding the use of $E = h\nu$ and $p = \frac{h}{\lambda}$. First, it is through them that the quantum character of the relations appears: $\Delta x\,\Delta\sigma \approx \Delta t\,\Delta\nu \approx 1$ are obtained from purely classical considerations. This is also obvious from the fact that it is the substitution of $E = h\nu$ and $p = \frac{h}{\lambda}$ which introduces h in the final result, i.e. in the UR. Second, Eq.(38) can be thought to express the principle of complementarity so dear to Bohr, with E and p regarded as having to do with the particle aspect of entities and ν and σ the wave one.

5.3.2 Uncertainty Relations, vacuum fluctuations and virtual particles

As briefly discussed in introduction, in NRQM a particle in a “harmonic oscillator” energy well has a ground state, “zero-point” energy of $\frac{1}{2}\hbar\omega$, and this is generally seen as a consequence of the UR for position and momentum, which implies corresponding URs for potential and kinetic energies respectively.
One derivation relating the UR to the “zero-point” energy goes as follows. If one considers the mechanical energy of the particle and:
- one substitutes the “momentum uncertainty” Δp for p in the kinetic energy term and the “position uncertainty” Δx for x in the potential energy specific

[65] “Addition in Proof” to [26]; [21], p.24. The thought experiment in question involves determining the position of an electron by scattering a photon off it, thereby leading to a discontinuous change in the momentum of the electron.

[66] A wave packet is a superposition of waves of different frequencies, hence of different wave numbers.

to a harmonic oscillator,[67]
- one makes use of the minimal form of the UR $\Delta x \Delta p = \frac{\hbar}{2}$, substituting for Δp,
- one minimizes the resulting expression for the mechanical energy with respect to Δx,

one finds that this energy has a global minimum (at $\Delta x = \sqrt{\frac{\hbar}{2m\omega}}$), and that its value there is $\frac{1}{2}\hbar\omega$.[68]
Note that no assumption has been made regarding the exact meaning of Δx and Δp, i.e. whether they should be viewed as representing experimental inaccuracies due to the limitations of instruments, statistical spreads such as standard deviations, or are associated with wave packets.
This derivation would seem to motivate an interpretation of the UR that goes beyond an epistemic one, whereby the relations would be understood as limiting only our knowledge of the position and momentum.[69] That is, if we interpret the zero-point energy result to mean that a particle in its ground state *actually has* an energy of $\frac{1}{2}\hbar\omega$, this would imply that the UR tell us something about the value a variable really has. How exactly this would come about seems less clear. One account would be to say that the uncertainty in the UR arises because of an intrinsic indeterminacy in the variables at stake, i.e. that there is truly no fact of the matter as to how much potential(/kinetic) energy the particle has exactly; that therefore the potential and kinetic energies cannot be both actually and exactly zero, hence their sum, the total energy, cannot either. In this story, ironically, the degree of indeterminacy in the component energies determines the exact value of their sum. This would correspond to an ontological interpretation of the UR as discussed by Hilgevoord and Uffink in connection to Heisenberg's ideas, since the potential and kinetic energies would have to be unsharp in such a sense.
If instead one views (Δx) and (Δp) as standard deviations, one can restrict oneself to the minimal interpretation of the UR, which is only epistemic. This would imply that the $\frac{1}{2}\hbar\omega$ value for the zero-point energy would have to be interpreted in a similar way.

We just considered zero-point energies in NRQM, we now turn back to QFT, and the half-quanta of its vacuum field. As stated previously, these are what are termed "vacuum fluctuations", and they are often described as evanescent "virtual particles" that borrow from the vacuum enough energy to become real for an amount of time too short for them to be observed, courtesy of the UR between time and energy.[70]

[67] Taking the ground state energy to be: $E_0 = \frac{1}{2m}(\Delta p)^2 + \frac{1}{2}m\omega^2(\Delta x)^2$.

[68] Note that this derivation relies on using equality in the UR, and that this is possible here because we are dealing with an harmonic oscillator potential: the wave function of the particle in such a potential is a gaussian, which constitutes a case of minimum uncertainty.

[69] Hence of the potential and kinetic energies.

[70] It may be worth stressing that this use of the phrase "virtual particles" differs from the standard, formal use of it in the context of Feynman diagrams notably: there "virtual particles" are particles that are "off shell", meaning that they are not obliged to satisfy the relativistic energy-momentum relation. What the two definitions have in common is that in

Now whether we are justified to talk about zero-point energies in terms of "fluctuations", which suggests a dynamical process, and whether we have ground to accept the "virtual particles" account are two different issues. Yet they are related: the latter requires the former, since "virtual particles" require energy to vary with time by small amounts, in turn increasing and decreasing — i.e. they require energy to fluctuate.

What makes the issue of whether this account is correct a fascinating one are its implications for the meaning of the UR. These play a double role in the scheme, because the uncertainty in energy and the uncertainty in time play different ones. For the energy uncertainty, the UR have to concern a matter of fact about the world, and to tell us something about the properties of entities — as could already be said in the context of zero-point energies in NRQM. So it requires the UR to be ontological in the sense used by Hilgevoord and Uffink. But it could be argued that now, in addition, it is ontological in a further sense: this "uncertainty" is responsible for the very existence of some entities. Indeed the scheme describes virtual particles to actually *exist* because they acquire enough energy to become full quanta courtesy of the UR. The reason the implications of zero-point energies in NRQM are not as extreme is because the latter cannot describe particle creation, so the quanta concern the energy of a particle that NRQM has to take as pre-existing. In QFT by contrast, the very *existence* of an entity (a particle) depends on the *value of a property* of another (the field): the energy quanta pertain to the field, and not to a particle from the outset; whether one is there or not depends on the value of a property of this field, i.e. its energy. While the energy is a full quantum, a particle actually exists. Its virtual character consists in its inability to maintain this existence long enough to be observed, which brings us to the second role played by the UR.

While the uncertainty in energy is responsible for the "effect" itself (i.e. the existence of the particles), the uncertainty in time provides an explanation for what could otherwise be deemed a contradiction with experimental facts, i.e. our inability to observe it. Hence in this account the uncertainty in time is both about a physical fact (the particles exist for a short time), and a limit on our knowledge (virtual particles are unobservable as a matter of principle).

However, there are serious issues with the "virtual particles" account.

Indeed, what part the energy half quanta are thought to play in this description is not clear. Their relation to the concepts of vacuum fluctuations and virtual particles certainly seems to be mediated by the UR: virtual particles are explicitly said to be the direct result of time-energy UR, fluctuations are certainly ascribed to uncertainty (although exactly how is again unclear), and field half quanta are thought to be due to the UR by analogy to the zero-point energy of the NRQM harmonic oscillator.[71] This analogy is suggested by the fact that both the NRQM particle and the QFT electromagnetic field have non-zero ground state energies as a result of operators not commuting. In this context the field and canonical momentum of QFT are the analogs of the NRQM posi-

both cases, the particles are said to be unobservable as a matter of principle.

[71] Particle in an harmonic oscillator potential well.

tion and momentum respectively. However there are issues with this analogy.
A first issue is that what is needed for the "virtual particles" account to make sense is an analog of the NRQM energy-time UR, not position-momentum. To what extent this is problematic is questionable: being relativistic, QFT puts position and time on the one hand, and energy and momentum on the other on the same footing. Hence quantum field and canonical momentum are both functions of time and position. Since their commutation relation involves time as much as it does position, one could argue that the field-canonical momentum commutation relation is the relevant one. Furthermore, it has been argued that "in the context of special relativity the energy-time form might be thought of as a consequence of the position-momentum version, because x and t (or rather, ct) go together in the position-time four vector." ([28], p.112.) So the field-canonical momentum relation could be said to stand in the same relation to both the energy-time and the position-momentum URs.[72] However it is unclear how the "virtual particles" account should be phrased to reflect the field-canonical momentum relation.
A second problem with the "virtual particles" account, and with even merely talking of fluctuations is that they implicitly invoke a dynamical process. Although the field-canonical momentum commutation relation indirectly involves time, it can hardly be said to express dynamical evolution. The operators may be time dependent, but these commutation relations are usually considered at equal times, so it is difficult to see how an uncertainty in time would arise — never mind how that would then yield a dynamical account. On the other side of the correspondence, as noted by Robert Klauber, being energy eigenstates half quanta do not sit well with a dynamical story either:

> Those half quanta appear to simply be steadily "sitting" in the vacuum, and not "popping in and out" of it. There is no apparent mechanism whereby they exist part of the time, but not all of the time.[73]

Partly related to the latter is a third, and perhaps more worrisome issue — which too would undermine not only an account in terms of virtual particles, but even the soundness of speaking of fluctuations. There exists an important disanalogy between the NRQM commutation relation and its QFT counterpart: unlike position and momentum in NRQM, quantum fields and their conjugate momenta are not observable, in that their expectation value is zero. For this reason, Klauber has argued that the UR one can associate with this commutation relation is in fact "meaningless" ([29], p.273).[74]
So all in all, it is hard to see how the commutation relations responsible for the half quanta in QFT are related to the time-energy UR invoked in the "virtual

[72] even though only position-momentum is based on commutation relations.
[73] [29], p.270. Klauber's work is primarily a textbook on QFT but some parts of it, notably his chapter 10 entitled "The vacuum revisited" is a critical reflection on the uses of the phrase "virtual particles".
[74] The same remarks apply to the commutation relation between creation and annihilation operators.

particles" account — and perhaps required to speak of fluctuations too.

Can the time-energy UR nevertheless be used to justify this account, independently of any reference to the field-canonical momentum commutation relations? After all, both Heisenberg's and Bohr's views on the UR did not *require* commutation relations.[75] As we saw, Bohr even provided a derivation of the position-momentum as well as the time-energy UR without using commutation relations. And above all, the time-energy UR are not associated with a commutation relation anyway.
This approach may be more hopeful, but even in this case it is not clear how a dynamical account could emerge. One may seek inspiration in the fact that the time-energy UR has been used to account for the period of oscillation of a system between its stationary states: the energy difference between two eigenstates is related to the period of oscillation by this UR. However this requires the system to be in a superposition of the two energy eigenstates, which a field in the vacuum state by definition is not. Another suggestive account is the relationship between the lifetime of a particle type and the spread found in its mass (i.e. rest energy) through repeated measurements, which is also given by the time-energy UR. However in this account what the UR controls is the time it takes the particles to cease to exist; by contrast what we need as a matter of priority for the "virtual particles" account is for it to provide them with the required energy to become full quanta in the first place. Still, it is a suggestive story.

Yet a different approach would be to seek dynamics without invoking the UR for that purpose. This would not provide much ground for the virtual particles account, but would presumably justify the use of the term "fluctuations". In this respect the FDT could be helpful, since it has been interpreted to show that in the $T = 0$ limit, the fluctuating force is due to the vacuum field.[76] This certainly would seem to require that the latter fluctuates, in the same, dynamical sense as the Planck radiation field does in the high temperature limit. Generally speaking, in applications of the FDT, zero-point effects and high-temperature limit, classical effects stand in the same relation to the fluctuating force, each contributing a term to it. It therefore seems reasonable that if thermal effects give rise to a fluctuating, dynamical force, so do the zero-point effects.
In so far that the associated zero-point field can be said to originate from the UR (which in itself is questionable), this would indeed seem to justify interpreting the latter in more than an epistemic sense but in an ontological one as well, i.e. as actually affecting the value of variables. This conclusion would seem to hold irrespective of whether these fluctuations are thought to require the pre-existence of the radiation field or not, since the UR does imply a zero-point term in both cases.

[75]The relation between UR and commutation relations were introduced by Kennard, as discussed above.

[76]i.e. the zero-point term in Eq.(15) for $E(\omega, T)$ manifests itself as fluctuations.

6 Conclusions

The effects traditionally ascribed to vacuum fluctuations can also be ascribed to a source field, and we showed how this underdetermination comes about: the operator relative to the system commutes with the operator for the total field (vacuum + source), hence derivations yield the same results irrespective of the ordering chosen, but it does not commute with the operator for the source field nor for the vacuum field, so different orderings yield different relative contributions from these two fields. We saw that the ambiguity can be lifted by demanding that only Hermitian operators be used throughout the derivation, which corresponds to choosing symmetric ordering, and implies that both fields contribute — hence that the vacuum field has an observable effect. This demand is justified on the ground that symmetric ordering is more physical, in so far that it involves Hermitian operators throughout the calculations. However, it implies regarding the positive (or negative) frequency part of the field as unphysical.

At the same time, physicists have often appealed to the Fluctuation-Dissipation Theorem when discussing the relative contributions of source and vacuum fields. Although the FDT paints a clear intuitive physical picture in the high temperature, classical limit, its interpretation in the $T = 0$ limit of zero-point contributions is not as straightforward. If we take it to mean that the fluctuating force it describes is exerted by the same entity as the dissipative force, then we should conclude that it is due to a source field. More often than not however, the FDT has been thought to suggest that the fluctuating force is due to the vacuum field. Either way the FDT requires the presence of a source field for dissipative effects to take place, and what we mean by vacuum field in such situation requires some care: in QFT the phrase refers to the electromagnetic field being in its ground state. The very presence of a source field precludes this, so what is at stake are then fluctuations in the (excited) field having the same origin as the zero-point energy of the vacuum field — i.e. the Uncertainty Relations.

Most physicists have interpreted the FDT as evidence for contributions from vacuum fluctuations. But this view has been heavily criticized at some point, notably by Edwin Jaynes, who interpreted the FDT to "show that source field effects are the same as if vacuum fluctuations were present" — i.e. that the fluctuating force is exerted by the same field as the dissipative one. He argued that people incorrectly assumed vacuum fluctuations to be responsible because the expression for the spectral energy density of the vacuum field appeared in their result, when in fact this expression could just as well be due to the source field.

However Jaynes' warning against unduly interpreting some expressions as representing vacuum field effects does not hold for the research involving operator ordering. Nor does it seem to undermine an argument in favor of vacuum field effects regarding the consistency of the quantum theory of radiation: taking the vacuum field into account in the Heisenberg equation of motion for an accelerating charge is necessary for the position-momentum commutation relation to

hold.
Therefore, at this point it seems that the vacuum field does contribute to the physical effects in question.
Whether describing the role of the vacuum field as involving fluctuations in a dynamical sense, or to speak of such "vacuum fluctuations" as evanescent, virtual, particles that exist courtesy of the UR, is far more questionable. The best reason we have to speak of fluctuations seems to be the FDT, which does refer to a fluctuating force. High temperature limit, classical contributions to this force are certainly dynamical in character, which would suggest their zero-point counterparts are as well.
Furthermore, in so far that this zero-point field can be said to originate from the UR, it suggests that in this context at least, the latter should be interpreted in more than a minimal, epistemic sense, but in an ontological one as well.

7 Appendix: Underdetermination and Ordering of operators

First case: Q given by PE

We denote by Q the operator for the quantity of interest, the field operator by E and the operator for the quantity that describes the atomic system by P. If:

$$Q = PE, \tag{40}$$

and we refrain from making any assumption regarding the ordering of E and P within Q, the most general form the latter can take can be written:

$$Q = \lambda PE + (1-\lambda)EP, \tag{41}$$

where λ is arbitrary. This implies that the contributions to Q due to the vacuum and source fields, i.e. respectively , Q_0 and Q_S can in turn be written:

$$Q_0 = \lambda PE_0 + (1-\lambda)E_0P; \quad Q_S = \lambda PE_S + (1-\lambda)E_SP. \tag{42}$$

P, E_S and E_0 are Hermitian, so their conjugates are:

$$P = P^*; \quad E_0 = E_0^*; \quad E_S = E_S^*, \tag{43}$$

and therefore the Hermitian conjugates of Q^S and Q_0^* are:

$$Q_0^* = \lambda E_0^* P^* + (1-\lambda)P^*E_0^* = \lambda E_0 P + (1-\lambda)PE_0, \tag{44}$$

$$Q_S^* = \lambda E_S^* P^* + (1-\lambda)P^*E_S^* = \lambda E_S P + (1-\lambda)PE_S. \tag{45}$$

In order for Q_0 and Q_S to be hermitian, the following needs to be satisfied:

$$Q_0 = Q_0^*; \quad Q_S = Q_S^* \tag{46}$$

$$\lambda PE_0 + (1-\lambda)E_0P = \lambda E_0P + (1-\lambda)PE_0; \quad \lambda PE_S + (1-\lambda)E_SP = \lambda E_SP + (1-\lambda)PE_S \tag{47}$$

$$\lambda = 1-\lambda \Rightarrow \lambda = \frac{1}{2} \tag{48}$$

That is, Q_0 and Q_S hermitian implies that $\lambda = \frac{1}{2}$. In this case the two operators take the form:

$$Q_0 = \frac{1}{2}PE_0 + \frac{1}{2}E_0P; \quad Q_S = \frac{1}{2}PE_S + \frac{1}{2}E_SP \tag{49}$$

This corresponds to the following expression for Q:

$$\begin{aligned} Q &= Q_0 + Q_S \\ &= \frac{1}{2}(PE_0 + PE_S) + \frac{1}{2}(E_0P + E_SP) \\ &= \frac{1}{2}PE + \frac{1}{2}EP. \end{aligned} \tag{50}$$

If we now expand the field operators in terms of their positive and negative frequency components:

$$Q = \frac{1}{2}PE + \frac{1}{2}EP = \frac{1}{2}P(E^+ + E^-) + \frac{1}{2}(E^+ + E^-)P, \tag{51}$$

which corresponds to a choice of symmetric ordering of the field operators, as announced.[77]

Second case (two-level atom): Q given by terms of the form $E^-P^- + E^+P^+$

Now Q is no longer by PE but by a sum over terms of the form:

$$E^-P^- + E^+P^+, \tag{52}$$

where E^- and P^- can be ordered arbitrarily since they commute, and the same holds for E^+ and P^+. If we then choose to order them differently in both terms, i.e.:

$$E^-P^- + P^+E^+ \quad \text{or} \quad P^-E^- + E^+P^+ \tag{53}$$

we notice that we get back these two expressions in the same order when taking their hermitian conjugates:

$$(E^-P^- + P^+E^+)^\dagger = P^+E^+ + E^-P^- = E^-P^- + P^+E^+ \tag{54}$$

$$(P^-E^- + E^+P^+)^\dagger = E^+P^+ + P^-E^- = P^-E^- + E^+P^+ \tag{55}$$

If we now want to express this in terms of the separate contributions Q_0 and Q_S of the vacuum and source fields, the same structure is preserved as we substitute $E^+ = E_0^+ + E_S^+$ and $E^- = E_0^- + E_S^-$. That is, for instance:

$$Q_0^\dagger = (E_0^-P^- + P^+E_0^+)^\dagger = P^+E_0^+ + E_0^-P^- = E_0^-P^- + P^+E_0^+ = Q_0. \tag{56}$$

So we see that with both ordering choices in Eq.(53), Q_0 and Q_S are hermitian. Yet these two choices are not equivalent to one another: they constitute different choices of ordering that lead to different relative contributions from the vacuum and source fields.

In such a situation, i.e. when the variable of interest Q is given by a sum over terms of the form Eq.(52), removing the ordering freedom requires imposing an additional requirement, in addition to demanding that Q_0 and Q_S be hermitian. One needs to impose also that the expression for Q be rewritten in a form that involves products of only E with P,[78] not products of either E^+ or E^- with P^+ or P^- as above. The ordering that corresponds to this requirement is again the symmetric ordering.[79]

[77] Recall that normal ordering consists in placing annihilation operators to the right of creation operators, anti-normal ordering the is the reverse, and symmetric ordering the linear combination of both in equal proportions.

[78] Or products of appropriate combinations of $E^+ - E^-$ and $P^+ - P^-$.

[79] These derivations are clarified versions of what can be found in [8].

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