Unitary Inequivalence in Classical Systems

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Abstract

I provide an algebraic formulation of classical field theories and use this to probe our interpretation of algebraic theories more generally. I show that the *problem of unitarily inequivalent representations*, as discussed in Ruetsche (2011), arises in classical theories just as in quantum theories, and I argue that this gives reason to not be a Hilbert Space Conservative when interpreting algebraic theories.

1 Introduction

Ruetsche (2011) argues that a *problem of unitarily inequivalent representations* arises in quantum theories with infinitely many degrees of freedom. When one attempts to "quantize" a classical theory, i.e. formulate a quantum theory for the same system the classical theory was meant to describe, one is not guaranteed that the resulting quantum theory is unique.[1] For a classical theory with infinitely many degrees of freedom, e.g. a field theory or a statistical theory in the thermodynamic limit, there are many inequivalent theories which compete to be called its "quantization".

Work on the problem of unitarily inequivalent representations is done in the algebraic framework for quantum theories. One begins by representing physical observables as elements of an abstract C*-algebra, which captures the structure that all of the representations of the canonical commutation or anti-commutation relations have in common. One then looks for concrete representations of that algebra in the bounded operators on some Hilbert space. The problem of unitarily inequivalent

[1] This assumes that a quantum theory is a concrete Hilbert space representation of canonical commutation or anti-commutation relations. This assumption is not universally shared and we will see later on how it is challenged.

representations forces us to make a choice in interpreting the algebraic formalism. Either one takes the algebraic formulation as basic, and becomes an *Algebraic Imperialist*, or else one privileges a particular Hilbert space representation as a *Hilbert Space Conservative*.[2]

This paper investigates the algebraic formalism itself for the purpose of assessing these interpretive options. I will step back from the specific details of quantum theory, whose understanding is already so controversial, and shift to the simpler context of classical field theory. As is already known,[3] one can use the very same algebraic formalism previously mentioned to describe classical theories as well as quantum ones. Since the interpretation of classical field theory is at the very least better understood and better agreed upon than that of quantum mechanics, this suggests that we can use classical field theory to probe our understanding of the abstract algebraic formalism. This paper uses an algebraic reformulation of classical field theory as a concrete case in order to investigate whether similar interpretive and foundational issues arise in the classical case as in the quantum case, and whether natural solutions are suggested in the classical case.

There is a tradition in philosophy of physics, specifically in the philosophy of space and time, of translating our previous theories into the language of our current theories for the purpose of assessing what precisely is novel about our new theories. For example, many use Newton-Cartan theory (geometrized Newtonian gravitation) as a way of translating classical Newtonian gravitation into a framework in which one can compare it with general relativity. Upon doing so, one finds that at least some of the things that have been said were distinctive features of general relativity turn out to be features of Newton-Cartan theory as well (see, e.g. Weatherall 2011). The purpose of this paper is to make a similar point about the relationship between quantum field theory and classical field theory. While many have argued that the problem of unitarily inequivalent representations is a conceptual problem for quantum theories with infinitely many degrees of freedom, I will argue that the mathematical features that lead to this problem are not distinctive of quantum theories. Furthermore, I will argue that looking at the classical case helps us understand just what the problem is and how to go about looking for solutions.

[2] I take this terminology from Ruetsche (2011, Ch. 6), who adapts it from Arageorgis (1995).

[3] It has for some time now been accepted that abelian algebras may be used to represent the observables of a classical system (e.g. Summers & Werner 1987, 2441), but it was not until recently that such a formulation was made explicit (Brunetti et al. 2012).

I will show that the problem of unitarily inequivalent representations arises in the algebraic formulation of classical field theories just as it arises in quantum theories.[4] More specifically, I will show that in classical theories, the GNS representations for any two distinct pure states are unitarily inequivalent. In a certain sense, the problem is even more pressing in the classical case. However, there is another sense in which it is not a problem at all in the classical case because its solution is obvious. I will argue that the position of Hilbert Space Conservatism (as extended to classical field theories) is untenable, so the obvious solution to the problem is to not be a Hilbert Space Conservative. Furthermore, I will show that the standard argument against Algebraic Imperialism fails in the classical context. To the extent that this moral concerning the algebraic formalism carries over from the classical case to the quantum case, one also ought not be a Hilbert Space Conservative about algebraic quantum theory.[5] The mathematical results displayed in my argument are not new, but their significance seems to have been overlooked. I believe it is worthwhile to present them in this new light if only because they help us to understand issues that have previously been puzzling to some of those attempting to understand the algebraic formalism.

2 Preliminaries

2.1 Algebraic Quantum Field Theory

Here we introduce the basic concepts of algebraic quantum field theory.[6] One begins with a net of algebras, which is an association of a unital C*-algebra $\mathfrak{A}(O)$, called a *local algebra of observables*, to each suitable[7] open bounded region O of some spacetime $\mathcal{M}$

$$O \mapsto \mathfrak{A}(O)$$

[4]By this, I do not mean to assert that unitarily inequivalent representations in classical theories bear the same interpretation outlined above. In particular, I do not mean that there is some kind of non-uniqueness in our description of classical theories or that we have competing versions of classical theories. Rather, I mean that analogous technical results hold in the classical case that force us to reconsider our interpretive options.

[5]In this paper I will not examine the arguments against Algebraic Imperialism in the quantum case, which some might see as arguments in favor of Hilbert Space Conservatism. See Feintzeig (2015) for an examination of those arguments.

[6]For more on operator algebras, see Kadison & Ringrose (1997) and Sakai (1971). For more on the algebraic formalism and axioms of algebraic quantum field theory, see Halvorson (2006) and Ruetsche (2011, ch. 4).

[7]For example, one might restrict attention to only open double cones (see, e.g. Halvorson 2006, 740)

This net of C*-algebras bears the following initial interpretation: each self-adjoint element of $\mathfrak{A}(O)$ corresponds to a possible local observable, a quantity measurable within O. This net of algebras is required to satisfy a number of general axioms, including

Isotony: If $O_1 \subseteq O_2$, then $\mathfrak{A}(O_1) \subseteq \mathfrak{A}(O_2)$.

Isotony guarantees that the net $\mathfrak{A}(O)$ has an inductive limit $\mathfrak{A}$, called the *quasilocal algebra*:

$$\mathfrak{A} = \overline{\bigcup_{O \subseteq \mathcal{M}} \mathfrak{A}(O)}$$

A *state* is a positive, normalized linear functional ω on $\mathfrak{A}$. A state ω bears the following initial interpretation: for each self-adjoint $A \in \mathfrak{A}$, the number $\omega(A)$ is the expectation value for a measurement of the observable corresponding to A in the state ω.

Importantly, this algebraic formalism translates back into the familiar Hilbert space theory once we are given a state. A *representation* of a C*-algebra $\mathfrak{A}$ is a pair $(\pi, \mathcal{H})$, where $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is a *-homomorphism into the bounded linear operators on some Hilbert space $\mathcal{H}$. One of the most fundamental results in the theory of C*-algebras, known as the GNS theorem, asserts that for each state ω on $\mathfrak{A}$, there exists a representation $(\pi_\omega, \mathcal{H}_\omega)$ of $\mathfrak{A}$, known as the *GNS representation for* ω, and a (cyclic) vector $\Omega_\omega \in \mathcal{H}_\omega$ such that for all $A \in \mathfrak{A}$,

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle$$

One may find representations of $\mathfrak{A}$ on different Hilbert spaces, and in this case one wants to know when these can be understood as "the same representation". This notion of "sameness" is given by the concept of unitary equivalence:[8] two representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are *unitarily equivalent* if there is a unitary mapping $U : \mathcal{H}_1 \to \mathcal{H}_2$ which intertwines the representations, i.e. for each $A \in \mathfrak{A}$,

$$U\pi_1(A) = \pi_2(A)U$$

[8]See Ruetsche (2011, Ch. 2.2) and Clifton & Halvorson (2001, Sec. 2.2-2.3) for more on unitary equivalence as a notion of "sameness of representations."

The specified unitary mapping U sets up a way of translating between density operator states on $\mathcal{H}_1$ and density operator states on $\mathcal{H}_2$, and between observables in $\mathcal{B}(\mathcal{H}_1)$ and observables in $\mathcal{B}(\mathcal{H}_2)$. The GNS representation for a state ω is unique in the sense that any other representation $(\pi, \mathcal{H})$ of $\mathfrak{A}$ containing a cyclic vector corresponding to ω is unitarily equivalent to $(\pi_\omega, \mathcal{H}_\omega)$.

What has been said until now has not depended at all on the structure associated with the spacetime $\mathcal{M}$. Now we restrict attention to a spacetime with an affine structure. For our purposes, this could be either relativistic Minkowski spacetime or classical Newtonian spacetime. All that matters is that we have an associated vector space G, which can be thought of as the translation group of that spacetime. To make the algebraic structure of the theory compatible with this affine structure of the spacetime, one employs a further axiom:[9]

> **Translation Covariance:** There is a representation $x \mapsto \alpha_x$ of the translation group G into the automorphism group of $\mathfrak{A}$ such that for all $O \subseteq \mathcal{M}$
>
> $$\alpha_x(\mathfrak{A}(O)) = \mathfrak{A}(O + x)$$

Translation covariance tells us that translating our algebra via the automorphism α_x will get us the same result as translating our region of spacetime and looking at the algebra associated with the new region. A *vacuum state*[10] is a state ω_0 on the quasilocal algebra $\mathfrak{A}$ which is invariant under the translation group, i.e. for each $x \in G$, $\omega_0 \circ \alpha_x = \omega_0$. One often works in a *vacuum representation* $(\pi_{\omega_0}, \mathcal{H}_{\omega_0})$, the GNS representation of $\mathfrak{A}$ in a chosen vacuum state ω_0.

2.2 Algebraic Classical Field Theory

We now apply the concepts and constructions of the algebraic formalism to classical field theories. This is an attempt to translate classical field theory into a framework in which it can be compared with quantum field theory. We do *not* claim that the algebraic formalism provides the

[9]For more on Translation Covariance and translation-invariant states, see Ruetsche (2011, 105-106) and Halvorson (2006, Sec. 2.2).

[10]For the purposes of this paper, I take this as a definition of vacuum states. One might insist that translation-invariance is merely a necessary condition for being a vacuum state. If so, one can simply substitute "translation-invariant" every time I say "vacuum".

best or most useful way of understanding classical field theory for all purposes, or even that it captures the full, rich structure of the classical theory. As such, one might worry that the algebraic setting is artificial in the classical case, and that we might want to interpret the formalism differently here than in quantum theories. I will have more to say about the differences between the classical and quantum case below, but for now I note that all I require for my arguments is that quantum and classical theories share a background of very weak interpretive assumptions—namely, that observables correspond to quantities that we can measure of our system and states assign expectation values to those observables.

This weak interpretive background has made the practice of translating classical theories into the algebraic language common and fruitful among researchers investigating the boundary between the classical and the quantum. For example, Summers & Werner (1987) prove algebraic results about the status of the Bell inequalities when classical and quantum systems are coupled, and Landsman (1998, 2006) reviews techniques for understanding algebraic classical theories as the appropriate limits of quantum theories, e.g. using deformation quantization. In order for this practice to make sense, we need to be able to understand classical and quantum theories in a unified framework. Doing so will give us the tools we need here to compare classical field theory with quantum field theory and in particular to analyze the significance of unitarily inequivalent representations.

We restrict ourselves for simplicity and concreteness to the algebraic formulation of the classical theory of a real scalar field[11] $\varphi : \mathcal{M} \to \mathbb{R}$. Each such smooth field $\varphi \in C^\infty(\mathcal{M})$ represents a possible configuration of our system. Thus, we take an appropriate subcollection[12] $\mathcal{U} \subseteq C^\infty(\mathcal{M})$ (say, of solutions to some partial differential equation) to be our configuration space. Observables in this theory will be functions $f : \mathcal{U} \to \mathbb{C}$ (Landsman, 1998, 1). Each such observable f corresponds to a possible measurable quantity of the system, and $f(\varphi)$ is the value that we would measure if the

[11]One could generalize by repeating these constructions for smooth sections of an arbitrary vector bundle over $\mathcal{M}$.

[12]Some technical caveats about "appropriate" configuration spaces: first, we require that $\mathcal{U}$ be closed under translations (see footnote 15). Second, we also would like to put a topology on $\mathcal{U}$ so that we can restrict attention to observables that are continuous fundtions on $\mathcal{U}$ (see footnote 14).

actual field configuration were φ. The observables form a C*-algebra with operations defined by

$$(fg)(\varphi) := f(\varphi) \cdot g(\varphi)$$
$$(f+g)(\varphi) := f(\varphi) + g(\varphi)$$
$$f^*(\varphi) := \overline{f(\varphi)}$$

for all $f, g : \mathcal{U} \to \mathbb{C}$.[13]

One may pick out the local observables (which form a C*-algebra with the operations defined above suitably restricted) to a given region $O \subseteq \mathcal{M}$, and thereby define the net of observables, according to the following rule:

Definition 1. *(Classical Net) An observable $f : \mathcal{U} \to \mathbb{C}$ is in $\mathfrak{A}(O)$, and said to be* local *to O, iff for all $\varphi, \varphi' \in \mathcal{U}$ such that $\varphi_{|\overline{O}} = \varphi'_{|\overline{O}}$, we have $f(\varphi) = f(\varphi')$.*

This definition fits with our intuitive understanding of what it means for an observable to be local—an observable f is local to O just in case the value of f on any field configuration φ depends only on the values φ takes within O (or on the boundary of O). The following proposition shows that one can take the inductive limit of the classical net to obtain the classical quasilocal algebra $\mathfrak{A}$.

Proposition 1. *The classical net satisfies Isotony, i.e. if $O_1 \subseteq O_2$, then $\mathfrak{A}(O_1) \subseteq \mathfrak{A}(O_2)$.*

Proof. Suppose $O_1 \subseteq O_2$ and choose any $f \in \mathfrak{A}(O_1)$. For any $\varphi, \varphi' \in \mathcal{U}$, if $\varphi_{|\overline{O_2}} = \varphi'_{|\overline{O_2}}$, then $\varphi_{|\overline{O_1}} = \varphi'_{|\overline{O_1}}$, and therefore $f(\varphi) = f(\varphi')$, which shows that $f \in \mathfrak{A}(O_2)$. □

One can also define a representation of the translation group α_x on $\mathfrak{A}$ (via an intermediary group β_x acting on $\mathcal{U}$) as follows. For each $x \in G$, define $\beta_x : \mathcal{U} \to \mathcal{U}$ by

$$\beta_x(\varphi)(p) := \varphi(p + x)$$

[13] If one wishes, one may restrict attention to the continuous functions with respect to some appropriate topology. For example, one may use the compact-open topology or the Whitney topology (Brunetti et al. 2012). Furthermore, one must restrict attention to bounded functions so that the supremum norm is well defined everywhere on the algebra.

for all $\varphi \in \mathcal{U}$ and $p \in \mathcal{M}$.[14] Then for each $x \in G$, define $\alpha_x : \mathfrak{A} \to \mathfrak{A}$ by

$$\alpha_x(f)(\varphi) := f(\beta_x(\varphi))$$

for all $f \in \mathfrak{A}$ and $\varphi \in \mathcal{U}$.

Proposition 2. *The classical net satisfies Translation Covariance, i.e.* $f \in \mathfrak{A}(O)$ *iff* $\alpha_x(f) \in \mathfrak{A}(O + x)$.

Proof. Suppose $f \in \mathfrak{A}(O)$. Then given any two $\varphi, \varphi' \in \mathcal{U}$, if $\varphi_{|\overline{O+x}} = \varphi'_{|\overline{O+x}}$, then for any $p \in O$,

$$\beta_x(\varphi)(p) = \varphi(p + x) = \varphi'(p + x) = \beta_x(\varphi')(p)$$

and so $\beta_x(\varphi)_{|\overline{O}} = \beta_x(\varphi')_{|\overline{O}}$. It follows that

$$\alpha_x(f)(\varphi) = f(\beta_x(\varphi)) = f(\beta_x(\varphi')) = \alpha_x(f)(\varphi')$$

which shows that $\alpha_x(f) \in \mathfrak{A}(O + x)$. The other direction follows similarly. □

Given this representation of the translation group, we can identify vacuum states. Each determinate field configuration $\varphi \in \mathcal{U}$ defines a state ω_φ on $\mathfrak{A}$ by

$$\omega_\varphi(f) := f(\varphi)$$

for all $f \in \mathfrak{A}$. If φ is a *constant* determinate field configuration, then it is invariant under the translation group β_x, because for all $p \in \mathcal{M}$, $\beta_x(\varphi)(p) = \varphi(p + x) = \varphi(p)$. It follows that ω_φ is invariant under the translation group α_x, because for all $f \in \mathfrak{A}$,

$$\omega_\varphi(\alpha_x(f)) = \alpha_x(f)(\varphi) = f(\beta_x(\varphi)) = f(\varphi) = \omega_\varphi(f)$$

[14] For this definition to make sense, we must require that $\mathcal{U}$ is closed under translations, i.e. if $\varphi \in \mathcal{U}$, then $\beta_x(\varphi) \in \mathcal{U}$.

Thus any constant determinate field configuration defines a vacuum state on the quasilocal algebra. One intuitive choice of vacuum state corresponds to the determinate constant field configuration $\varphi_0 = 0$. Just as in algebraic quantum field theory, once one has chosen a vacuum state by choosing a constant field configuration $\varphi \in \mathcal{U}$, one can work in its corresponding vacuum representation. We shall investigate the status of this vacuum representation, and the GNS representations for all states defined by determinate field configurations in the next section.

3 Unitary Inequivalence

3.1 Two Possible Interpretations

Quantizing a classical theory involves two steps: first, one isolates from the observables of the classical theory certain algebraic relations (usually commutation or anti-commutation relations) and second, one finds a representation of the resulting algebra in the bounded operators on some Hilbert space. In the case where the original classical theory has finite degrees of freedom, the Stone-von Neumann theorem (Ruetsche 2011, 41, Clifton & Halvorson 2001, 427) shows that all of the Hilbert space representations we end up with are unitarily equivalent and so the resulting quantum theory is unique.[15] However, it is well known that theories with infinite degrees of freedom, including field theories and statistical theories in the thermodynamic limit, violate the assumptions of the Stone-von Neumann theorem. Even though the first step of the quantization procedure yields a unique algebra for the quantum theory, this algebra admits many unitarily inequivalent representations when we attempt to perform the second step (Ruetsche 2011, Ch. 3.3). Because of this, it appears that many theories of interest for physics do not have a unique quantization.

Anyone hoping to understand quantum theories of infinite systems, like quantum field theory, must decide which quantum theory to consider. One has two options to choose from.[16] First, one can be an *Algebraic Imperialist* by asserting that a quantum theory is given in full by the abstract

[15] The Stone-von Neumann theorem carries additional assumptions as well. It assumes that the phase space of the classical theory is symplectic and that the representation is continuous in an appropriate sense.

[16] For more on these positions and their advantages and disadvantages, see Arageorgis (1995) and Ruetsche (2002, 2003, 2006, 2011 Ch. 6). Of course, as Ruetsche describes, there are many more subtle interpretive options, but we deal here only with two of the simplest cases. Ruetsche herself holds neither view described here.

algebra of observables and the states on that algebra rather than its Hilbert space representations. The abstract algebra captures a structure that all Hilbert space representations have in common, so the Algebraic Imperialist chooses to focus only on this structure. To do so is to proclaim that all the work that has been done on interpreting the Hilbert space formalism for ordinary quantum mechanics with finite degrees of freedom cannot yield a complete and adequate interpretation for the case of infinite degrees of freedom. According to the Algebraic Imperialist, "the extra structure one obtains along with a concrete representation of [$\mathfrak{A}$] is extraneous." (Ruetsche 2011, 132). All that matters is the abstract algebraic structure. The physically measurable quantities are given by the observables (self-adjoint elements) in $\mathfrak{A}$, and the physically possible states are given by the states on $\mathfrak{A}$.

On the other hand, if one wants to be a *Hilbert Space Conservative* and maintain an interpretation via the Hilbert space formalism like those usually discussed for ordinary quantum mechanics, then one must pick a particular Hilbert space representation to interpret. When one is working in the context of a particular Hilbert space $\mathcal{H}$, one can define the weak operator topology on $\mathcal{B}(\mathcal{H})$. The weak operator topology is defined by the following criterion for convergence (Halvorson 2006, 732, Ruetsche 2011, Ch. 4): a net $\{A_i\}_{i\in\mathcal{X}}$ converges in the weak operator topology to A (written $A_i \to A$) just in case for all $\phi, \psi \in \mathcal{H}$,

$$\langle \phi, A_i\psi\rangle \to \langle \phi, A\psi\rangle$$

Recall that the GNS theorem allows us to take any C*-algebra of observables $\mathfrak{A}$ and, having chosen some state ω, represent it via the representation π_ω as a subalgebra $\pi_\omega(\mathfrak{A}) \subseteq \mathcal{B}(\mathcal{H}_\omega)$ for some Hilbert space $\mathcal{H}_\omega$. Using the weak operator topology on $\mathcal{B}(\mathcal{H}_\omega)$ as the physically relevant notion of approximation,[17] one can include in any algebra of observables the operators that are physically indistinguishable from or well approximated by the observables already picked out in our algebra.

[17] The motivation for this standard practice is that in the weak operator topology, a sequence of observables well approximates (i.e. converges to) another observable just in case it approximates it with respect to all possible expectation values and transition probabilities, and hence with respect to the empirical predictions of the theory. I believe there is room to question already whether this motivation succeeds in justifying the weak operator topology (Feintzeig 2015), but one need not do so for the argument against Hilbert Space Conservatism that follows.

To do so, we take the weak operator closure (written $\overline{\pi_\omega(\mathfrak{A})}$) of our original algebra of observables $\pi_\omega(\mathfrak{A})$. If the state ω that we used to take our GNS representation is pure,[18] then, because the representation π_ω is irreducible,[19] it follows that $\overline{\pi_\omega(\mathfrak{A})} = \mathcal{B}(\mathcal{H}_\omega)$ (see Sakai 1971, Prop. 1.21.9, 52). So taking the GNS representation for a pure state and closing in the weak operator topology brings us back to the familiar situation where our observables are *all* of the bounded self-adjoint operators on a Hilbert space.[20]

For the Hilbert Space Conservative, one such particular Hilbert space representation of the abstract algebra specifies the physical possibilities. Having chosen a pure state ω on $\mathfrak{A}$ and obtained its GNS representation $(\pi_\omega, \mathcal{H}_\omega)$, the Hilbert Space Conservative follows the standard practice in ordinary quantum mechanics (for finite degrees of freedom). For the Hilbert Space Conservative, the physically measurable quantities are the self-adjoint elements of $\overline{\pi_\omega(\mathfrak{A})} = \mathcal{B}(\mathcal{H}_\omega)$, and the physically possible states are density operators on $\mathcal{H}_\omega$.

3.2 Classical Analogue

I will show that unitarily inequivalent representations are endemic to classical field theories (considered prior to any quantization procedure) and that the proper understanding of them provides an argument against Hilbert Space Conservatism. Here, I do not mean to propose a complete interpretation of classical field theories or an adequate notion of physical equivalence for them; I confine myself to displaying technical results about the existence of unitarily inequivalent representations and using them to argue against the Hilbert Space Conservative. As we will see, in a certain sense unitarily inequivalent representations are more problematic for classical field theories than quantum theories because, while in the quantum case at least the GNS representations for *some* pure states are unitarily equivalent, in the classical case the GNS representations of any two

[18] A state ω is *pure* if whenever $\omega = a_1\omega_1 + a_2\omega_2$ for states ω_1, ω_2, it follows that $\omega_1 = \omega_2 = \omega$.

[19] A representation $(\pi, \mathcal{H})$ of $\mathfrak{A}$ is *irreducible* if the only subspaces left invariant under the action of $\pi(\mathfrak{A})$ are $\{0\}$ and $\mathcal{H}$.

[20] There is another interpretive stance that one might take if one wanted to use the tools of Hilbert space theory for physics. One might allow for reducible representations and then use either the *universal representation* or the *reduced atomic representation* of the algebra, both of which are formed by taking direct products of GNS representations (see Kadison & Ringrose 1997, p. 281). These representations lead to a position along the lines of what Ruetsche calls *Universalism* (Ruetsche 2011, p. 145). In this paper, I will deal only with the Hilbert Space Conservative that is committed to irreducible representations; see Feintzeig (2015) for a discussion of Universalism.

distinct pure states are *always* unitarily *in*equivalent. This section assumes the setting of classical field theories set out in section 2.2.

Proposition 3. *Let $\varphi \in \mathcal{U}$ be a field configuration. Then the corresponding state ω_φ (defined in section 2.2) is pure.*

Proof. The state ω_φ is multiplicative: if $f, g \in \mathfrak{A}$, then

$$\omega_\varphi(fg) = (fg)(\varphi) = f(\varphi) \cdot g(\varphi) = \omega_\varphi(f) \cdot \omega_\varphi(g)$$

It follows immediately that ω_φ is a pure state (Kadison & Ringrose 1997, 269, Thm. 4.4.1). □

In particular, any vacuum state corresponding to a constant determinate field configuration is pure. The rest of the argument follows merely from the fact that the quasilocal algebra $\mathfrak{A}$ is abelian.

Proposition 4. *Let $\mathfrak{A}$ be an abelian C*-algebra. Let ω be a pure state on $\mathfrak{A}$ and let $(\pi_\omega, \mathcal{H}_\omega)$ be the GNS representation of $\mathfrak{A}$ for ω. Then $\mathcal{H}_\omega$ is one dimensional.*

Proof. Because ω is pure, the GNS representation $(\pi_\omega, \mathcal{H}_\omega)$ for the state ω is irreducible (Kadison & Ringrose 1997, 728, Thm. 10.2.3). Since $\mathfrak{A}$ is abelian, any irreducible representation of $\mathfrak{A}$ is on a one-dimensional Hilbert space (Kadison & Ringrose 1997, 744). □

This means that the GNS representation for any pure state, and thus for the state corresponding to any determinate field configuration, is so weak that it has the power to represent *only a single state* as a density operator.

Proposition 5. *Let ω_1 and ω_2 be distinct pure states on an abelian C*-algebra $\mathfrak{A}$. Let $(\pi_{\omega_1}, \mathcal{H}_{\omega_1})$ and $(\pi_{\omega_2}, \mathcal{H}_{\omega_2})$ be the GNS representations of $\mathfrak{A}$ for the states ω_1 and ω_2 with corresponding cyclic vectors Ω_{ω_1} and Ω_{ω_2}, respectively. Then $(\pi_{\omega_1}, \mathcal{H}_{\omega_1})$ and $(\pi_{\omega_2}, \mathcal{H}_{\omega_2})$ are unitarily inequivalent.*

Proof. (See Kadison & Ringrose 1997, 744) Suppose there is a unitary transformation $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that for all $A \in \mathfrak{A}$,

$$U\pi_{\omega_1}(A) = \pi_{\omega_2}(A)U$$

Then since the Hilbert spaces are one-dimensional, it follows that $U\Omega_{\omega_1} = e^{i\theta}\Omega_{\omega_2}$ for some $\theta \in \mathbb{R}$. So for all $A \in \mathfrak{A}$

$$\begin{aligned}
\omega_1(A) = \langle \Omega_{\omega_1}, \pi_{\omega_1}(A)\Omega_{\omega_1}\rangle &= \langle \Omega_{\omega_1}, U^{-1}\pi_{\omega_2}(A)U\Omega_{\omega_1}\rangle \\
&= \langle U\Omega_{\omega_1}, \pi_{\omega_2}(A)e^{i\theta}\Omega_{\omega_2}\rangle \\
&= \langle e^{i\theta}\Omega_{\omega_2}, e^{i\theta}\pi_{\omega_2}(A)\Omega_{\omega_2}\rangle \\
&= e^{-i\theta}e^{i\theta}\langle \Omega_{\omega_2}, \pi_{\omega_2}(A)\Omega_{\omega_2}\rangle = \langle \Omega_{\omega_2}, \pi_{\omega_2}(A)\Omega_{\omega_2}\rangle = \omega_2(A)
\end{aligned}$$

Therefore, $\omega_1 = \omega_2$. □

Thus, if determinate field configurations $\varphi, \varphi' \in \mathcal{U}$ determine distinct states ω_φ and $\omega_{\varphi'}$, then, since the quasilocal algebra $\mathfrak{A}$ of any classical field theory is abelian, the corresponding GNS representations for ω_φ and $\omega_{\varphi'}$ will be unitarily inequivalent. In particular, the vacuum representation mentioned above for the vacuum state determined by the constant zero field configuration is unitarily inequivalent to the GNS representation for any distinct state. This is, in a certain sense, worse than the quantum case because while in quantum field theory a representation may include many states as density operators, in the classical case, each representation includes only a single state.

In another sense, however, unitarily inequivalent representations are less problematic because we have no prior reason to regard them as rival theories. One might think it was never a good idea to look for Hilbert space representations of classical theories in the first place. We will examine the differences between the classical and the quantum case in the next section, but first we will look in more detail to see why these unitarily inequivalent representations arise in the classical case.

The quasilocal algebra $\mathfrak{A}$, since it is abelian, is *-isomorphic to $C(\mathcal{P}(\mathfrak{A}))$, the continuous functions on the compact Hausdorff space $\mathcal{P}(\mathfrak{A})$ of pure states of $\mathfrak{A}$ with the weak*-topology (Kadison & Ringrose 1997, 270, Thm 4.4.3). (Recall that each determinate field configuration defines one of these pure states.) As such, each observable $f \in \mathfrak{A}$ corresponds to a function $\hat{f} \in C(\mathcal{P}(\mathfrak{A}))$ defined by

$$\hat{f}(\omega) = \omega(f)$$

for each pure state $\omega \in \mathcal{P}(\mathfrak{A})$. Taking the GNS representation for a pure state ω amounts to choosing a measure on the space $\mathcal{P}(\mathfrak{A})$ (See Kadison & Ringrose 1997, 744; Landsman 1998, 55), which defines an (L^2) inner product, hence constructing a Hilbert space as follows. By the Riesz-Markov theorem (Reed & Simon 1980, 107, Thm. IV.14), each pure state ω on $\mathfrak{A}$ corresponds to a unique regular Borel measure μ_ω on $\mathcal{P}(\mathfrak{A})$ such that for all $f \in \mathfrak{A}$

$$\omega(f) = \int_{\mathcal{P}(\mathfrak{A})} \hat{f} d\mu_\omega$$

The GNS representation of $\mathfrak{A}$ for the pure state ω is unitarily equivalent to the representation[21] $(\pi_\omega, \mathcal{H}_\omega)$ on the Hilbert space $\mathcal{H}_\omega = L^2(\mathcal{P}(\mathfrak{A}), d\mu_\omega)$, with π_ω defined by

$$\pi_\omega : f \mapsto M_{\hat{f}}$$

where the operator $M_{\hat{f}}$ is defined as multiplication by the function $\hat{f}$, i.e. for any $\psi \in \mathcal{H}_\omega$,

$$M_{\hat{f}}\psi = \hat{f} \cdot \psi$$

Furthermore, when ω is pure,

$$\omega(f) = \hat{f}(\omega) = \int_{\mathcal{P}(\mathfrak{A})} \hat{f}\delta(\omega)$$

where $\delta(\omega)$ is the point mass or delta function centered on ω. It follows by the uniqueness clause of the Riesz-Markov theorem that $d\mu_\omega = \delta(\omega)$. Now every vector $\psi \in \mathcal{H}_\omega$ is defined by a single complex number—the value of ψ on $\omega \in \mathcal{P}(\mathfrak{A})$—which shows that $\mathcal{H}_\omega$ is one-dimensional (Prop. 4). Each observable $f \in \mathfrak{A}$ is represented on this Hilbert space via π_ω as the value that $\hat{f}$ takes at ω, i.e. the complex number $\hat{f}(\omega) = \omega(f)$. Notice that the representation π_ω will in general not be faithful, i.e. not one-to-one, because multiple observables may be assigned the same expectation value by the state ω. Choosing a distinct pure state with which to take the GNS representation amounts to

[21] Here, the relevant cyclic vector Ω_ω is the constant unit function.

choosing a distinct measure on $\mathcal{P}(\mathfrak{A})$, which means that some observable will be represented as a different complex number, and hence the representations will be unitarily inequivalent (Prop. 5).

I believe these propositions yield a straightforward argument against the Hilbert Space Conservative in the classical case. The argument begins from a single, extremely weak premise: every classical field theory admits multiple distinct solutions to its governing equations of motion, yielding multiple possible determinate field configurations and hence, multiple distinct states. Each GNS representation for a pure state on $\mathfrak{A}$, because it is one-dimensional, has only the resources to represent a single state as a density operator; any other state can only be represented as a density operator on the Hilbert space of a unitarily *in*equivalent (irreducible) representation of the algebra. By focusing only on one particular representation, the Hilbert Space Conservative would have to deny the existence of multiple distinct states, which is absurd.

We see that the problem for the Hilbert Space Conservative is that they do not have access to resources powerful enough to represent all of the states that are physically possible or physically significant, and the classical case illustrates this emphatically. The Algebraic Imperialist, on the other hand, always has access to all of the states on the algebra. The obvious solution to the problem of unitarily inequivalent representations for the classical case is to not limit oneself to a particular irreducible representation, because no representation will suffice for representing more than a single state.

It is worth noting the broader significance of unitarily inequivalent representations in the classical case beyond the Conservative-Imperialist debate; this will help us to understand the role that a representation is playing in the theory. Whenever two states give rise to unitarily inequivalent irreducible GNS representations, one says that there is a *superselection rule* between those two states (see Earman 2008). One way of explaining the physical significance of this statement is that when there exists a superselection rule between two states, they cannot be coherently formed into a superposition. For example, there is a superselection rule in quantum mechanics between states with integer and half-integer angular momentum. In the classical case, the fact that the GNS representations of any distinct pure states are unitarily inequivalent implies that, in the same sense, there is a superselection rule between *any* distinct pure states. This corresponds to the fact that classical physics does not allow for coherent superpositions, or in other words that superposition is a strictly quantum phenomenon (Landsman 1991, 5354).[22] The above propositions show a sense in which one can *derive* this fact—that classical and quantum physics differ in allowing coherent superpositions—from the basic structure of the relevant algebras of observables.

The classical case helps us to see that a representation is a tool that focuses in on some particular states while ignoring others. Namely, a representation focuses in on a set of states that can be formed into coherent superpositions. In the quantum case, because there are nontrivial superpositions, we end up with representations on a nontrivial Hilbert space; but in the classical case, because there are no superpositions, taking a representation focuses us on only a single state. Since in both cases representations are giving us physical information about when superpositions can be formed, this gives us reason to believe that representations have a similar physical significance in the classical and the quantum case—it just so happens that classical states behave differently than quantum ones in that they cannot be formed into superpositions. The above results then show that we should not think of a representation as a full theory as the Hilbert Space Conservative would have us do. Rather, we should think of a representation as some part of the theory containing only a subcollection of all of the states the theory deems possible—namely, a subcollection of states that can be coherently superposed with each other.

3.3 Are quantum theories different?

Even if Conservativism is untenable in the classical case, one might still contend that the quantum case is sufficiently different that Conservatism is tenable there. For example, one might point to the fact that the algebras of observables used in quantum theories are typically *simple*, i.e. they have no non-trivial two-sided ideals.[23] This implies that every irreducible representation of the algebra is faithful, which is in stark contrast to the classical case in which (as we saw above) every irreducible representation fails to be faithful because the Hilbert space is one-dimensional. Hence, one might contend that even if my argument is sound in the classical case, the quantum case is

[22] Superselection sectors are also sometimes thought to have some extra dynamical significance. In the classical case, unitarily inequivalent representations are only significant for the notion of superposition and not for dynamics.

[23] A *two-sided ideal* is a subalgebra $\mathfrak{I} \subseteq \mathfrak{A}$ such that for all $A \in \mathfrak{I}$ and $B \in \mathfrak{A}$, we have $AB \in \mathfrak{I}$ and $BA \in \mathfrak{I}$.

different enough that the argument will not apply.

The quantum case is certainly different in many respects, but (as mentioned in section 2.2) we have reason to want to understand classical and quantum theories as being part of a unified framework, as many of the practitioners of the theory do (Summers & Werner 1997; Landsman 2006). Insofar as we understand algebraic classical and quantum theories to fit into this unified framework with at least some basic shared interpretive assumptions, we can ask about the significance of certain mathematical operations that we can perform on both theories.

One might worry that the algebraic framework is not unified *enough* to allow for this kind of analysis. The pointwise multiplication operation of the classical algebra of observables leaves out much of the information encoded in the classical theory—specifically the dynamical information, which may be encoded (for example) in a Poisson bracket defined on the relevant phase space. On the other hand, in the quantum algebra, the noncommutative multiplication operation might be understood as encoding dynamical information because the Poisson bracket plays a fundamental role in defining the canonical commutation relations of the quantum theory. I think this is a real difference between the physical significance of the classical and quantum algebras, but it does not change the status of the arguments given in this paper. Here we have considered the unified algebraic framework only as a tool for representing the possible states and observables of the theory, without any consideration of the dynamics. *This is all the information we need* to answer questions that others have posed about physical equivalence and translatability between different representations (Clifton & Halvorson 2001; Ruetsche 2011; Baker 2011). The objection that classical multiplication does not contain dynamical information is irrelevant because none of the arguments here depend on dynamics at all.

Given that quantum and classical theories do fit into a unified framework, and given that this framework suffices for many scientific purposes, the analysis of the classical case at least shifts the burden of proof and leads one to question the Conservative's position. I believe it is absolutely obvious that in the classical case the Hilbert Space Conservative does not have access to enough states. This puts a burden on the Conservative even in the quantum case to show that she *can* represent enough states.

But when the Hilbert Space Conservative about quantum theories faces this question, we find that (regardless of the differences between the classical and the quantum) she cannot represent enough states in the quantum case. This claim is not new; Ruetsche (2002, 364; 2003, 1338; 2006, 480) argues at length for this conclusion, as follows. The GNS representations of certain physically significant pure states in quantum field theory and quantum statistical mechanics are unitarily inequivalent. For example, the GNS representations of states corresponding to different pure thermodynamic phases are unitarily inequivalent. This implies that neither of the Hilbert spaces of the individual representations can represent the other state as a density operator, and so neither Hilbert space can represent the other state as a physical possibility. A Hilbert Space Conservative, having chosen one particular pure state through which to consider the GNS representation, must deny that states which cannot be represented on the Hilbert space of that representation are physically possible. So, for example, a Hilbert Space Conservative can never judge different pure thermodynamic phases to be physically possible and hence can never give an explanation of phase transitions, which would require the coexistence of distinct phases (see Ruetsche 2002, 2003). The Hilbert Space Conservative cannot recover these physically significant explanations precisely because she cannot countenance all of the physically relevant states. As we have seen, this is exactly the same feature that leads to the downfall of Hilbert Space Conservatism in the classical case.

4 Conclusion

We have seen that classical theories can be formulated in the algebraic framework, and when we do so we find that the GNS representations of any two distinct pure states are unitarily inequivalent. The Hilbert Space Conservative uses only one of these GNS representations so she can only represent a single state as physically possible. Since all classical theories contain more than one state, this implies that the Hilbert Space Conservative cannot represent all possible states of the theory. This argument is exactly analogous to ones that Ruetsche gives in the quantum case (Ruetsche 2002, 2003, 2006, 2011), and I believe my arguments only bolster her conclusion. This is important because even though the arguments in the quantum case are well known, some (Baker 2011; Baker

& Halvorson 2013) continue to discuss Conservatism as if it were a viable option. Here, I hope to have shown, by consideration of the classical case, that Ruetsche's arguments against Hilbert Space Conservatism are already strong enough to defeat that position. Her arguments really do show, as is evident from the analogous argument in the classical case, that the Hilbert Space Conservative fails to represent all physically significant states, and it follows that Conservatism is untenable.

Even though I claim this argument rules out Hilbert Space Conservatism, it does not immediately lead one to Algebraic Imperialism. Ruetsche provides an important argument against Algebraic Imperialism, which leads her to an alternative "adulterated interpretation" (Ruetsche 2011). I will show briefly that this argument against Algebraic Imperialism does not apply in the classical case.

Ruetsche asserts that the Algebraic Imperialist does not have the resources to represent all of the physically possible observables. The Hilbert Space Conservative, having privileged some pure state ω and its GNS representation $(\pi_\omega, \mathcal{H}_\omega)$, acquires all of the observables (which Ruetsche terms *parochial observables*) in $\overline{\pi_\omega(\mathfrak{A})} = \mathcal{B}(\mathcal{H}_\omega)$. Many of these observables have real physical import (e.g., the temperature observable) but have no analogue in the abstract algebra. So the Hilbert Space Conservative gains access to more observables than the Algebraic Imperialist. These observables are physically significant, e.g. for giving explanations of thermodynamic phase transitions. According to Ruetsche, the Algebraic Imperialist runs into a problem because she cannot recognize these operators as physically possible observables, and so cannot vindicate such explanations.

However, the argument against Algebraic Imperialism does not apply in the classical case. In the classical case, no such parochial observables appear in the GNS representation of any pure state. Since the GNS representation $(\pi_\omega, \mathcal{H}_\omega)$ for any pure state ω is one-dimensional, and since $\pi_\omega(\mathfrak{A})$ contains the identity operator and is closed under scalar multiplication by complex numbers, it follows that $\pi_\omega(\mathfrak{A}) = \mathcal{B}(\mathcal{H}_\omega)$, and so the representation exhausts the possible observables on $\mathcal{H}_\omega$. In this case, $\overline{\pi_\omega(\mathfrak{A})} = \pi_\omega(\mathfrak{A})$, which means that there are no parochial observables. In other words every observable the Hilbert Space Conservative recognizes has an analogue in the abstract algebra, so the Algebraic Imperialist recognizes that observable too.[24]

The road is now paved to be an Algebraic Imperialist in the classical case. I believe that these

[24] Admittedly, this argument works *only for the classical case.* See Feintzeig (2015) for a general solution.

arguments help us understand the algebraic formalism more generally and that they push us toward Algebraic Imperialism in the quantum case as well. At the very least, the failure of Hilbert Space Conservatism that we've seen in this paper should force us to reconsider Algebraic Imperialism in the quantum case.

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