The Objective Indefiniteness Interpretation of Quantum Mechanics

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Draft (not for quotation)

May 28, 2013

Abstract

Quantum mechanics (QM) is not compatible with the usual Boolean logic of subsets where elements have definite properties "all the way down." But there is a recently developed dual logic of partitions (subsets and partitions are category-theoretically dual) which models indefinite elements that become more definite as distinctions are made. If quantum mechanics was *also* incompatible with this unique dual logic of partitions, then one might "press the panic button" and forsake realistic interpretation for an instrumentalist approach, postulate unknowable hidden variables to restore definiteness, or soar off into the void with "many worlds" or the like.

But quantum mechanics fits perfectly with the dual logic of partitions. There is no need for (more) bizarre flights of fancy to "interpret" QM. This objective indefiniteness approach to QM does not restore our common sense assumption of definiteness down into the quantum realm. But it does restore sanity and understanding to the whole framework. That is, we now have the logic that precisely fits QM—a logic that was developed independently (i.e., without any thought of a QM connection) and that is the unique mathematical dual to ordinary Boolean subset logic, the logic assumed in ordinary intuitions and in classical physics. Moreover the normalized counting measure on partitions gives the quantum-relevant logical information theory—just as Boole developed logical probabilities as the normalized counting measure on subsets. Indeed, when the mathematics of partition logic and logical information theory is linearized and lifted to complex vector spaces, then it yields the mathematical framework of quantum mechanics (but not the specifically physical postulates).

The key concepts explicated by this approach are the old ideas of "objective indefiniteness" (emphasized by Abner Shimony, Peter Mittelstaedt, and others), objective probabilities, and the objective realization of information, "its" from "dits" (= distinctions). This explicates the standard view that the description of a quantum state as a superposition is a complete description, i.e., that the indefiniteness of a superposition is objective.

Since partition logic, logical information theory, and the lifting program "derives" the mathematics of quantum mechanics, it shows how that QM framework can be interpreted—and this set of results gives what might be called the *objective indefiniteness interpretation* of quantum mechanics.

Contents

1 Introduction: the back story for objective indefiniteness

Classical physics is compatible with the common-sense view of reality that is expressed at the logical level in Boolean subset logic. Each element in the Boolean universe set is either definitely in or not in a subset, i.e., each element either definitely has or does not have a property. Each element is characterized by a full set of properties, a view that might be referred to as "definite properties all the way down."

It is now rather widely accepted that this common-sense view of reality is not compatible with quantum mechanics (QM). If we think in terms of only two positions, *here* and *there*, then in classical physics a particle is either definitely *here* or *there*, while in QM, the particle can be "neither definitely here nor there."[43, p. 144][1] This is not an epistemic or subjective indefiniteness of location; it is an ontological or objective indefiniteness. The notion of *objective indefiniteness* in QM has been most emphasized by Abner Shimony ([37],[38]).

> From these two basic ideas alone – indefiniteness and the superposition principle – it should be clear already that quantum mechanics conflicts sharply with common sense. If the quantum state of a system is a complete description of the system, then a quantity that has an indefinite value in that quantum state is objectively indefinite; its value is not merely unknown by the scientist who seeks to describe the system. ...Classical physics did not conflict with common sense in these fundamental ways.[37, p. 47]

Peter Mittelstaedt has also emphasized blurred, unsharp, or "incompletely determined" [33, p. 171] quantum states, and Shimony's and Mittelstaedt's students have continued this line of approach [4].[2]

Other philosophers have suggested related ideas; indeed the idea that a quantum state is in some sense "blurred" or "like a cloud" is now rather commonplace even in the popular literature. Indeed, it is the standard view that a description of a superposition quantum state is a *complete* description, i.e., the indefiniteness of a superposition state is in some sense objective.

Two questions arise:

- What is the logic of objective indefiniteness that plays the role analogous to Boolean subset logic (the logic of definite properties "all the way down")?

[1] This is usually misrepresented in the popular literature as the particle being "both *here* and *there* at the same time." Weinberg also mentions a particle "spinning neither definitely clockwise nor counterclockwise" and then notes that for elementary particles, "it is possible to have a particle in a state in which it is neither definitely an electron nor definitely a neutrino until we measure some property that would distinguish the two, like the electric charge."[43, pp. 144-145 (thanks to Noson Yanofsky for this reference)]

[2] See also the discussion by Falkenburg [15].

- And given such a logic, how would one fill in the gap between the austere level of logic and the rich mathematical framework of quantum mechanics?

These questions can now be answered. The logic of objective indefiniteness that plays the role analogous to subset logic is the recently developed dual logic of partitions.[14] Partition logic is not just "another alternative" logic; it is the unique mathematical dual (in the category-theoretic sense explained below) to subset logic. Hence if subset logic is not the logic for quantum mechanics, then the natural next step to check is if the dual logic of partitions fits QM. The point of this paper is to show that it fits perfectly.

Moreover, Boole developed logical finite probability theory based on the normalized counting measure on subsets [3], and the analogous theory based on the normalized counting measure of partitions is the logical version of information theory [13]. This logical information theory generalized to the density matrices of quantum mechanics precisely describes and interprets the changes made in a quantum measurement, and thus it is information theory relevant to QM.

The concepts and operations of partition logic and logical information theory are developed in the rather austere set-theoretic context; they needed to be "lifted" or linearized to the richer environment of vector spaces over the complex numbers. For instance, the partition join applied to a complete set of partitions defined on the same set to get the cardinality-one subsets–lifts to the partition join of the eigenspace partitions of a complete set of commuting operators (Dirac's CSCOs) to get the one-dimensional simultaneous eigenspaces. This lifting program is a ladder from sets to vector spaces, a conceptual "algorithm" that starts with the linearization map that takes a set U to the vector space $\mathbb{C}^U$. Going the other way down the ladder gives a "delifted" version of "quantum mechanics" on sets (i.e., in vector spaces over $\mathbb{Z}_2$) that represents the pure logic of QM with all the metrical aspects from the reals and complex numbers distilled out.

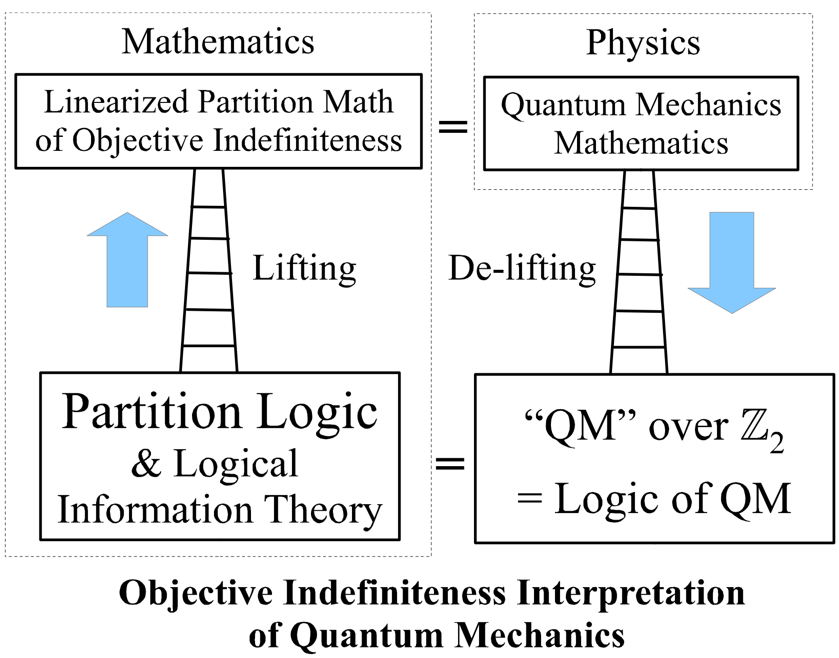

Figure 0: Objective indefiniteness research program

In Appendix 1, the lifting program is extended from representations of groups on sets to representations of groups on vector spaces over the complex numbers (where a group representation is seen as just a "dynamic" way to define an equivalence relation or partition). It has been shown by Jin-Quan Chen and the Nanjing School ([6], [7]) that the irreducible representations of symmetry groups can also be obtained by the partition join of the eigenspace partitions of a complete set of commuting operators (Chen's CSCOs).[3]

When applied to the concepts and operations of partition mathematics, the lifting program indeed yields the mathematics of quantum mechanics (both the Hilbert space mathematics and group representation theory). This strongly indicates that the vision of micro-reality provided by *the* dual form of logic (i.e., partition logic rather than subset logic) is, in fact, the micro-reality described by QM. Thus the development of the logic of partitions, logical information theory, and the lifting program provides the back story to the notion of objective indefiniteness. The result is the objective indefiniteness interpretation of quantum mechanics.

A final introductory point is to emphasize what parts of this "research programme" are simply mathematics as opposed physics. Partition logic and logical information theory are somewhat new mathematical theories. But the whole lifting program is just a repackaging or re-presentation of standard complex vector space and group theoretic mathematics to show that it is essentially a linearization over the complex numbers of the set-theoretical mathematics of partitions.[4] Partition logic, logical information theory, and the whole lifting program would still "be there" as mathematics even if the world was completely classical and there was no physically corroborated theory of quantum mechanics. In a classical world, the Boolean logic of subsets would give a proper logical description of entities that are definitely here or definitely there. The dual form of logic, partition logic, would just be a logical description of the subjective changes in information as distinctions are discovered; the objective interpretation would not apply physically. The lifting program would only be there as a piece of pure mathematics that essentially derives what *would* be a physical theory in that hypothetical world of objectively indefinite entities.

Now comes the physics. The 20^{th} century saw the development of quantum mechanics as a physical theory and revealed the inappropriateness of the Boolean logic of subsets for definite properties. This led to almost a century of increasingly bizarre "interpretations" of the mathematics of quantum mechanics. But with the development of partition logic, logical information theory, and the lifting program to the linearized partitional mathematics of vectors spaces and vector space representations of groups, it is seen that the mathematics of quantum mechanics *developed on physical grounds* is in fact the same as the partitional mathematics that describes a world of objectively indefinite entities. Since quantum mechanics is now a highly confirmed physical theory, this strongly suggests that the co-incidence between quantum mechanical mathematics and the lifted linearized mathematics of partitions is not just a coincidence. It strongly suggests that the underlying physical reality is one of objective indefiniteness.

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[3] "To use the eigen values of a complete set of commuting operators to character a quantum state vector is initiated by Dirac. The new group representation theory is a direct application of the Dirac method in the group representation space. Instead of the intricate methods of the traditional group representation theory, this approach provides a universal recipe for any group, i.e., provides a universal characterization of the IR [irreducible representations] and the calculation of the IB [irreducible basis] of any finite group in accordance with the concept and method of quantum mechanics." [42, p. 3]

[4] Since many set-theoretic operations can be interpreted as operations in linear vector spaces over $\mathbb{Z}_2$, the lifting program moves to the much richer base field of the complex numbers which introduces all the metrical aspects (as opposed to the simple zero-one aspects when $\mathbb{Z}_2$ is the base field).

2 The logic of partitions

2.1 From "propositional" logic to subset logic

Our treatment of partition logic here will only develop the basic concepts necessary to be lifted to the vector spaces of quantum mechanics.[5] But first it might be helpful to explain why it has taken so long for partition logic to be developed as the dual of subset logic, and to explain the duality.

George Boole [3] originally developed his logic as the logic of subsets. As noted by Alonzo Church:

> The algebra of logic has its beginning in 1847, in the publications of Boole and De Morgan. This concerned itself at first with an algebra or calculus of classes,... a true propositional calculus perhaps first appeared... in 1877.[8, pp. 155-156]

In the logic of subsets, a *tautology* is defined as a formula such that no matter what subsets of the given universe U are substituted for the variables, when the set-theoretic operations are applied, then the whole formula evaluates to U. Boole noted that to determine these valid formulas, it suffices to take the special case of $U = 1$ which has only two subsets $0 = \emptyset$ and 1 (or, equivalently, for any universe U, it suffices to consider the two subsets U and $\emptyset$). Thus what was later called the "truth table" characterization of a tautology was a theorem, not a definition.[6]

But over the years, the whole became identified with the special case. The Boolean logic of subsets was reconceptualized as "propositional logic." The truth-table characterization of a tautology became the *definition* of a tautology in propositional logic rather than a theorem in subset logic (see any textbook in logic). This facilitated the further analysis of the propositional atoms into statements with quantifiers and the development of model theory. But the restricted notion of "propositional" logic also had a downside; it hid the idea of a dual logic since propositions don't have duals.

Subsets and partitions (or equivalence relations or quotient sets) are dual in the category-theoretic sense of the duality between monomorphisms and epimorphisms. This duality is familiar in abstract algebra in the interplay of subobjects (e.g., subgroups, subrings, etc.) and quotient objects. William Lawvere calls the general category-theoretic notion of a subobject a *part*, and then he notes: "The dual notion (obtained by reversing the arrows) of 'part' is the notion of partition."[31, p. 85] The image of monomorphic or injective map between sets is a subset of the codomain, and dually the inverse-image of an epimorphic or surjective map between sets is a partition of the domain. The development of the dual logic of partitions was long delayed not because it was particularly difficult but because the very idea of a dual logic was not "in the air" due to the misconceptualization of subset logic as "propositional" logic.

2.2 Basic concepts of partition logic

Given a universe set U, a *partition* π of U is a set of non-empty subsets or blocks $\{B\}$ of U that are pairwise disjoint and whose union is U. In category-theoretic terms, a partition is a direct sum decomposition of a set, and that concept will lift, in the sets-to-vector-spaces lifting program, to the concept of a direct sum decomposition of a vector space.

[5] See [14] for a detailed development from the basic concepts up through the correctness and completeness theorems for a tableau system of partition logic.

[6] Alfred Renyi [36] gave a generalization of the theorem to probability theory.

In the Boolean logic of subsets, the basic algebraic structure is the Boolean lattice $\wp(U)$ of subsets of a universe set U enriched by the implication $A \Rightarrow B = A^c \cup B$ to form the Boolean algebra of subsets of U. In a similar manner, we form the lattice of partitions on U enriched by the partition operation of implication and other partition operations.

Given two partitions $\pi = \{B\}$ and $\sigma = \{C\}$ of the same universe U, the partition σ is *refined* by π, written by $\sigma \preceq \pi$, if for every block $B \in \pi$, there is a block $C \in \sigma$ such that $B \subseteq C$. Given the two partitions of the same universe, their *join* $\pi \vee \sigma$ is the partition whose blocks are the non-empty intersections $B \cap C$. In the lifting program below, the join of two partitions of the same set will lift to the join of two direct sum decompositions of a vector space that are in a certain sense "compatible."

The *top* of the lattice is the *discrete partition* $\mathbf{1} = \{\{u\} : u \in U\}$ whose blocks are all the singletons, and the *bottom* is the *indiscrete partition* (nicknamed the "blob") $\mathbf{0} = \{\{U\}\}$ whose only block is all of U. Together with the meet operation,[7] this defines the *lattice of partitions* $\prod(U)$ on U.[8]

We will use the representation of the partition lattice $\prod(U)$ as a lattice of subsets of $U \times U$. Given a partition $\pi = \{B\}$ on U, the *distinctions* or *dits* of π are the ordered pairs (u, u') where u and u' are in distinct blocks of π, and $\operatorname{dit}(\pi)$ is the *set of distinctions* or *dit set* of π. Similarly, an *indistinction* or *indit* of π is an ordered pair (u, u') where u and u' are in the same block of π, and $\operatorname{indit}(\pi)$ is the *indit set* of π. Of course, $\operatorname{indit}(\pi)$ is just the equivalence relation determined by π, and it is the complement of $\operatorname{dit}(\pi)$ in $U \times U$.

The complement of an equivalence relation is properly called a *partition relation* [also an "apartness relation"]. An equivalence relation is reflexive, symmetric, and transitive, so a partition relation is irreflexive [i.e., contains no self-pairs (u, u) from the diagonal Δ_U], symmetric, and anti-transitive, where a binary relation R is *anti-transitive* if for any $(u, u'') \in R$, and any other element $u' \in U$, then either $(u, u') \in R$ or $(u', u'') \in R$. Otherwise both pairs would be in the complement $R^c = U \times U - R$ which is transitive so $(u, u'') \in R^c$ contrary to the assumption.

Every subset $S \subseteq U \times U$ has a reflexive-symmetric-transitive *closure* $\overline{S}$ which is the smallest equivalence relation containing S. Hence we can define an *interior* operation as the complement of the closure of the complement, i.e., $\operatorname{int}(S) = \left(\overline{S^c}\right)^c$, which is the largest partition relation included in S. While some motivation might be supplied by thinking of the partition relations as "open" subsets and the equivalence relations as "closed" subsets, they do not form a topology. The closure operation is not a topological closure operation since the union of two closed subsets is not necessarily closed, and the intersection of two open subsets is not necessarily open.

Every partition π is represented by its dit set $\operatorname{dit}(\pi)$. The refinement relation between partitions, $\sigma \preceq \pi$ is represented by the inclusion relation between dit sets, i.e., $\sigma \preceq \pi$ iff $\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$.[9] The

[7] To define the meet $\pi \wedge \sigma$, consider an undirected graph on U where there is a link between any two elements $u, u' \in U$ if they are in the same block of π or the same block of σ. Then the blocks of $\pi \wedge \sigma$ are the connected components of that graph.

[8] For anything worthy to be called "partition logic," an operation of implication would be needed if not partition versions of all the sixteen binary subset operations. Given $\pi = \{B\}$ and $\sigma = \{C\}$, the *implication* $\sigma \Rightarrow \pi$ is the partition whose blocks are like the blocks of π except that whenever a block B is contained in some block $C \in \sigma$, then B is discretized, i.e., replaced by the singletons of its elements. If we think of a whole block B as a mini-$\mathbf{0}$ and a discretized B as a mini-$\mathbf{1}$, then the implication $\sigma \Rightarrow \pi$ is just the indicator function for the inclusion of the π-blocks in the σ-blocks. In the Boolean algebra $\wp(U)$, the set implication is related to the partial order by the relation, $A \Rightarrow B = U$ iff $A \subseteq B$, and we immediately see the corresponding relation for the partition implication, $\sigma \Rightarrow \pi = \mathbf{1}$ iff $\sigma \preceq \pi$, in the lattice of partitions $\Pi(U)$.

[9] Unfortunately in much of the literature of combinatorial theory, the refinement partial ordering is written the

join $\pi \vee \sigma$ is represented in $U \times U$ by the union of the dit sets, i.e., $\operatorname{dit}(\pi \vee \sigma) = \operatorname{dit}(\pi) \cup \operatorname{dit}(\sigma)$.[10] In this manner, the lattice of partitions $\prod(U)$ enriched by implication and other partition operations can be represented by the lattice of partition relations $\mathcal{O}(U \times U)$ on $U \times U$.

Representation	$\prod(U)$	$\mathcal{O}(U \times U)$
Partition	π	$\operatorname{dit}(\pi)$
Refinement order	$\sigma \preceq \pi$	$\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$
Top	$\mathbf{1} = \{\{u\} : u \in U\}$	$\operatorname{dit}(\mathbf{1}) = U \times U - \Delta_U$ all dits
Bottom	$\mathbf{0} = \{\{U\}\}$	$\operatorname{dit}(\mathbf{0}) = \emptyset$ no dits
Join	$\pi \vee \sigma$	$\operatorname{dit}(\pi \vee \sigma) = \operatorname{dit}(\pi) \cup \operatorname{dit}(\sigma)$
Meet	$\pi \wedge \sigma$	$\operatorname{dit}(\pi \wedge \sigma) = \operatorname{int}(\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma))$
Implication	$\sigma \Rightarrow \pi$	$\operatorname{dit}(\sigma \Rightarrow \pi) = \operatorname{int}(\operatorname{dit}(\sigma)^c \cup \operatorname{dit}(\pi))$
Any logical op. #	$\sigma \# \pi$	Int. of subset op. # applied to dit sets

Lattice of partitions $\prod(U)$ represented as lattice of partition relations $\mathcal{O}(U \times U)$.

2.3 Analogies between subset logic and partition logic

The development of partition logic was guided by some basic analogies between the two dual forms of logic. The most basic analogy is that a distinction or dit of a partition is the analogue of an element of a subset:

$$\boxed{u \text{ is an element of a subset } S \approx (u, u') \text{ is a distinction of } \pi}.$$

The top of the subset lattice is the universe set U of all possible elements and the top of the partition lattice is the partition $\mathbf{1}$ with all possible distinctions $\operatorname{dit}(\mathbf{1}) = U \times U - \Delta_U$ (all the ordered pairs minus the diagonal self-pairs which can never be distinctions). The bottoms of the lattices are the null subset $\emptyset$ of no elements and the indiscrete partition $\mathbf{0}$ of no distinctions. The partial orders in the lattices are the inclusion of elements $S \subseteq T$ and the inclusion of distinctions $\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$.

Intuitively, a *property* on U is something that each element has or does not have (like a person being female or not), while intuitively an attribute on U is something that each element has but with various values (like the weight or height of a person). The subsets of U can be thought of as abstract versions of properties of the elements of U while the partitions on U are abstract versions of the attributes on U where the different blocks of a partition represent the different values of the

other way around (so Gian-Carlo Rota sometimes called it "unrefinement"), and thus the "join" and "meet" are reversed, and the lattice of partitions is then "upside-down." That upside-down representation of the "lattice of partitions" uses the indit sets so it is actually the lattice of equivalence relations rather than the lattice of partition relations.

[10] But the intersection of two dit sets is not necessarily a dit set so to find the dit set of the meet $\pi \wedge \sigma$, we have to take the interior of the intersection of their dit sets, i.e., $\operatorname{dit}(\pi \wedge \sigma) = \operatorname{int}(\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma))$. These equations for the dit sets of the join and meet are theorems, not definitions, since the join and meet were already defined above. The general algorithm to represent a partition operation is to apply the corresponding set operation to the dit sets and then apply the interior to the result (if it is not already a partition relation). Thus, for instance,

$$\operatorname{dit}(\sigma \Rightarrow \pi) = \operatorname{int}(\operatorname{dit}(\sigma)^c \cup \operatorname{dit}(\pi)).$$

It is a striking fact (see [14] for a proof) that $\operatorname{int}(\operatorname{dit}(\sigma)^c \cup \operatorname{dit}(\pi))$ is the dit set of $\sigma \Rightarrow \pi$ previously defined as the indicator function for the inclusion of π-blocks in σ-blocks.

attribute. Technically, an *attribute* is given by a function $f : U \to \mathbb{R}$ (for some value set which we might take as the reals $\mathbb{R}$) and the partition induced by the attribute is the inverse image partition $\{f^{-1}(r) \neq \emptyset : r \in \mathbb{R}\}$. A real-valued attribute $f : U \to \mathbb{R}$ will lift to a Hermitian operator so that the attribute's inverse image partition $\{f^{-1}(r) \neq \emptyset : r \in \mathbb{R}\}$ lifts to the direct sum decomposition of the operator's eigenspaces, and the attribute's values r lift to the operator's eigenvalues.

	Subset Logic	Partition Logic
'Elements'	Elements $u \in U$	Distinctions $(u,u') \in U \times U - \Delta_U$
All 'elements'	Universe set U	Discrete partition **1** (all dits)
No 'elements'	Empty set $\varnothing$	Indiscrete partition **0** (no dits)
Duality	Subsets are images f() of injections $f: S \to U$	Partitions are inverse-images $f^{-1}()$ of surjections $f: U \to T$
Formula variables	Subsets of U	Partitions on U
Logical operations	$\cup$, $\cap$, $\Rightarrow$,...	Partition ops. = Interior of subset ops. applied to dit sets
Formula $\Phi(\pi,\sigma,...)$ holds at 'element'	Element u is in subset $\Phi(\pi,\sigma,...)$	Pair (u,u') is a distinction of partition $\Phi(\pi,\sigma,...)$
Valid formula $\Phi(\pi,\sigma,...)$	$\Phi(\pi,\sigma,...) = U$ for any subsets $\pi,\sigma,...$ of any U ($\lvert U\rvert \geq 1$)	$\Phi(\pi,\sigma,...) = \mathbf{1}$ for any partitions $\pi,\sigma,...$ on any U ($\lvert U\rvert \geq 2$)

Figure 1: Table of analogies between dual logics of subsets and partitions.

3 Logical information theory

We have so far made no assumptions about the finitude of the universe U. For a finite universe U, Boole developed the "logical" version of finite probability theory by assigning the normalized counting measure $\Pr(S) = \frac{|S|}{|U|}$ to each subset which can be interpreted as a probability under the Laplacian assumption of equiprobable elements. Using the elements-distinctions analogy, we can assign the analogous normalized counting measure of the dit set of a partition $h(\pi) = \frac{|\operatorname{dit}(\pi)|}{|U \times U|}$ to each partition which can be interpreted as the *logical information content* or *logical entropy* of the partition. Under the assumption of equiprobable elements, the logical entropy of a partition can be interpreted as the probability that two drawings from U (with replacement) will give a distinction of the partition.

	Logical Finite Prob. Theory	Logical Information Theory
'Outcomes'	Elements u∈U finite	Distinctions (u,u')∈U×U finite
'Events'	Subsets S ⊆ U	Dit sets dit(π) ⊆ U×U
Normalized counting measure	Prob(S) = |S|/|U| = logical probability of event S	h(π) = |dit(π)|/|U×U| = logical entropy of partition π
Equiprobable outcomes	Prob(S) = probability randomly drawn element is an outcome in S	h(π) = probability randomly drawn pair (w/replacement) is a distinction of π

Figure 2: Probability and information as normalized counting measures in dual logics

The probability of drawing an element from a block $B \in \pi$ is $p_B = \frac{|B|}{|U|}$ so the logical entropy of a partition can be written in terms of these block probabilities since $|\mathrm{dit}\,(\pi)| = \sum_{B \neq B' \in \pi} |B \times B'| = |U|^2 - \sum_{B \in \pi} |B|^2$. Hence:

$$h\,(\pi) = \frac{|\mathrm{dit}(\pi)|}{|U \times U|} = \frac{|U|^2 - \sum_{B \in \pi} |B|^2}{|U|^2} = 1 - \sum_{B \in \pi} p_B^2.$$

This formula has a long history (see [13]) and is usually called the *Gini-Simpson diversity index* in the biological literature [35]. For instance, if we partition animals by species, then it is the probability in two independent drawings that we will find animals of different species.

This version of the logical entropy formula also makes clear the generalization path to define the logical entropy of any finite probability distribution $p = (p_1, ..., p_n)$:

$$h\,(p) = 1 - \sum_i p_i^2.$$[11]

C. R. Rao [35] has defined a general notion of quadratic entropy in terms of a distance function $d(u, u')$ between the elements of U. In the most general "logical" case, the natural logical distance function is:

$$d\,(u, u') = 1 - \delta\,(u, u') = \begin{cases} 1 \text{ if } u \neq u' \\ 0 \text{ if } u = u' \end{cases}$$

and, in that case, the quadratic entropy is just the logical entropy.

Further details about logical information theory and the relationship with the usual notion of Shannon entropy can be found in [13]. For our purposes here, the important thing is the lifting of logical entropy to the context of vector spaces and quantum mathematics where for any density matrix ρ, the logical entropy $h\,(\rho) = 1 - \mathrm{tr}\left[\rho^2\right]$ allows us to directly measure and interpret the changes made in a measurement.

[11] In the general case, the p_i becomes a probability density function and the summation an integral.

4 Partitions and objective indefiniteness

4.1 Representing objective indistinctness

It has already been emphasized how Boolean subset logic captures at the logical level the common sense vision of reality where an entity definitely has or does not have any property, and has properties "all the way down."[12] We can now describe how the dual logic of partitions captures at the logical level a vision of reality with objectively indefinite (or indistinct)[13] entities.[14]

The key step is to:

interpret a subset S as a *single objectively indefinite element.*

To anticipate the lifted concepts in vector spaces, the fully distinct elements $\{u\} \subseteq U$ might be called "eigen-elements" and the single indistinct element S is a "superposition" of the eigen-elements $\{u\} \subseteq S$ (thinking of the collecting together $\{u, u', ...\} = S$ of the elements of S as their "superposition"). With distinctions, the indistinct element S might be refined into one of the eigen-elements $\{u\} \subseteq S$.

Abner Shimony ([37] and [38]), in his description of a superposition state as being objectively indefinite, adopted Heisenberg's [24] language of "potentiality" and "actuality" to describe the relationship of the eigenstates that are superposed to give an objectively indefinite superposition. This terminology could be adapted to the case of the sets. The elements $u \in S$ are "potential" in the objectively indefinite "superposition" S, and, with further distinctions, the indefinite element S might "actualize" to $\{u\}$ for one of the "potential" $u \in S$. Starting with S, the other $u \notin S$ are not "potentialities" that could be "actualized" with further distinctions.

This terminology is, however, somewhat misleading since the indefinite element S is perfectly actual; it is only the multiple eigen-elements $u \in S$ that are "potential" until "actualized" by some further distinctions. In a "measurement," a single actual indefinite element becomes a single actual definite element. Since the "measurement" goes from actual indefinite to actual definite, the potential-to-actual language of Heisenberg should only be used with proper care–if at all.

Consider a three-element universe $U = \{a, b, c\}$ and a partition $\pi = \{\{a\}, \{b, c\}\}$. The block $S = \{b, c\}$ is objectively indefinite between $\{b\}$ and $\{c\}$ so those singletons are its "potentialities" in the sense that a distinction could result in either $\{b\}$ or $\{c\}$ being "actualized." However $\{a\}$ is not a "potentiality" when one is starting with the indefinite element $\{b, c\}$.

Note that this objective indefiniteness is not well-described as saying that indefinite pre-distinction element is "simultaneously both b and c"; instead it is indefinite between b and c. That is, a "superposition" should *not* be thought of like a double exposure photograph which has two fully definite images (e.g., simultaneously a picture of say b and c). That imagery is a holdover

[12] The phrase "properties all the way down" means that two numerically distinct entities must differ by some property as in Leibniz's principle of the identity of indiscernables.

[13] The adjectives "indefinite" and "indistinct" will be used interchangeably as synonyms. The word "indefiniteness" is more common in the QM literature, but "indistinctness" has a better noun form as "indistinctions" (with the opposite as "distinctions").

[14] An indefinite entity, like a block in a partition, is *only* defined by the distinctions that have been made; there are no other properties "all the way down." Hence two such objectively indefinite entities defined by the same distinctions (i.e., in the "same state") are objectively indistinguishable in the sense familiar from quantum statistics. This role of partitions in group representation theory is treated in the first appendix.

from classical wave imagery (e.g., in Fourier analysis) where definite eigen-waveforms are superposed to give a superposition waveform. Instead of a double-exposure photograph, a superposition representation might be thought of as "a photograph of clouds or patches of fog." (Schrödinger quoted in: [22, p. 66])[15] Regardless of the (imperfect) imagery, one needs some way to indicate what are the definite eigen-elements that could be "actualized" from a single indefinite element S, and that is the role in the set case of conceptualizing $S = \{u, u', ...\}$ as a collecting together or a "superposition" of certain "potential" eigen-elements $u, u', ... \in U$.

The following is another attempt to clarify the imagery.

Eigenstate 1: Guy Fawkes with goatee	
Eigenstate 2: Guy Fawkes with mustache	
Objectively indistinct state before (facial hair) distinctions were made is the pre-distinction state.	
But objectively indistinct state may be represented by superposition of possible distinct alternatives: $\vert\text{goatee}\rangle + \vert\text{mustache}\rangle$	

Figure 3: Indistinct pre-distinction state *represented* as superposition of potential post-distinction states

This means, for instance, that one should not think of a particle's "wave function" $\psi(x, y, z)$ as being "spread out in space" over everywhere that $|\psi(x, y, x)|^2 > 0$; instead the particle's position is objectively indefinite between all those positions in space where it might be found with probability density $|\psi(x, y, x)|^2$ under a position measurement.

The following table gives yet another attempt at visualization by contrasting a classical picture and an objectively indefinite (or "quantum") picture of a "particle" getting from A to B.

[15] Schrödinger distinguishes a "photograph of clouds" from a blurry photograph presumably because the latter might imply that it was only the photograph that was blurry while the underlying objective reality was sharp. The "photograph of clouds" imagery for a superposition connotes a clear and complete picture of an objectively "cloudy" or indefinite reality.

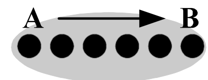
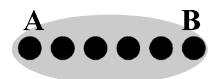
Classical trajectory from A to B.	
Subjective indefiniteness of classical position ("cloud of ignorance").	
Quantum trajectory: like definte position at A, indefinite position in transition, and definite position at B.	
Particle with objectively indefinite location...	
may be *represented* as superposition of possible eigen-positions.	

Figure 4: Getting from A to B in classical and quantum ways

The classical trajectory is a sequence of definite positions. A state of subjective indefiniteness is compatible with a classical trajectory when we have a "cloud of ignorance" about the actual definite location of the particle. The "quantum trajectory" might be envisaged in terms of "flights and perchings"[16] as starting with a definite "perch" or location at A, then evolving to an objectively indefinite state (with the various positions as potentialities), and then finally another "look" or measurement that achieves a definite "perch" at location B. The particle in its objectively indefinite position state is represented as the superposition of the possible definite position states; it does not traverse the intermediate definite positions in order to get from A to B.[17]

4.2 The conceptual duality between the two lattices

The conceptual duality between the lattice of subsets and the lattice of partitions could be described (following Heisenberg) using the rather meta-physical notions of substance and form. Consider what happens when one starts at the bottom of each lattice and moves towards the top.

[16]The "flights and perchings" metaphor is from William James [28, p. 158] and according to Max Jammer, that description "was one of the major factors which influenced, wittingly or unwittingly, Bohr's formation of new conceptions in physics." [29, p. 178] As David Hawkins put it, the picture at the quantum level "is no longer that of a steady deterministic flow..., but that of states and transitions, 'flights and perchings,' in which the perchings are more stable and flights more abrupt than classical ideas would have allowed." [23, p. 198]

[17]In terms of "hawks and hounds" imagery, the hound covers definite positions between A and B but the hawk flies from perch A to perch B without having definite ground positions in between. We might imagine a high wall between A and B with two slits in it. The hound would have to go through slit 1 or slit 2 to get from A to B; the hawk would not. But that is only a crude metaphor. The hawk has a definite position in a third dimension, but there is no indication that an indefinite position on the two-dimensional plane is well-represented by a definite position in a "higher" dimension.

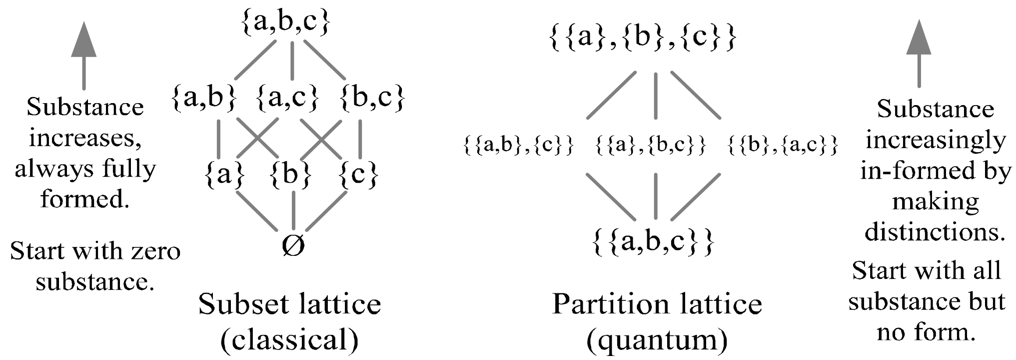


Figure 5: Conceptual duality between the two lattices

At the bottom of the Boolean lattice is the empty set $\emptyset$ which represents no substance. As one moves up the lattice, new elements of substance with fully definite properties are created until finally one reaches the top, the universe U. Thus new substance is created but each element is fully formed and distinguished in terms of its properties.

At the bottom of the partition lattice is the blob $\mathbf{0}$ which represents all the substance but with no distinctions to in-form the substance. As one moves up the lattice, no new substance is created but distinctions objectively in-form the indistinct elements as they become more and more distinct, until one finally reaches the top, the discrete partition $\mathbf{1}$, where all the eigen-elements of U have been fully distinguished from each other. Thus one ends up at the same place (macro-universe of fully distinguished elements) either way, but by two totally different but dual ways.

The notion of logical entropy expresses this idea of objective in-formation as the normalized count of the informing distinctions. For instance, in the partition lattice on a three element set pictured above, the logical entropy of the blob is always $h(\mathbf{0}) = 0$ since there are no distinctions. For a middle partition such as $\pi = \{\{a\}, \{b,c\}\}$, the distinctions are (a,b), (b,a), (a,c), and (c,a) for a total of 4 where $|U|^2 = 3^2 = 9$ so the logical entropy is $h(\pi) = \frac{|\text{dit}(\pi)|}{|U \times U|} = \frac{4}{9}$. For the discrete partition, there are all possible distinctions for a total of $|U|^2 - |\Delta_U| = 9 - 3 = 6$ so the logical entropy is $h(\mathbf{1}) = 1 - \frac{1}{|U|} = \frac{6}{9}$. In each case, the logical entropy of a partition is the probability that two independent draws from U will yield a distinction of the partition.

The progress from bottom to top of the two lattices could also be described as two creation stories.

- *Subset creation story*: “In the Beginning was the Void”, and then elements are created, fully propertied and distinguished from one another, until finally reaching all the elements of the universe set U.

- *Partition creation story*: “In the Beginning was the Blob”, which is an undifferentiated “substance,” and then there is a "Big Bang" where elements (“its”) are created by the substance being objectively in-formed (objective "dits") by the making of distinctions (e.g., breaking symmetries) until the result is finally the singletons which designate the elements of the universe U.[18]

[18] In his sympathetic interpretation of Aristotle’s treatment of substance and form, Heisenberg refers to the substance as: "a kind of indefinite corporeal substratum, embodying the possibility of passing over into actuality by means of the form."[24, p. 148] It was previously noted that Heisenberg’s "potentiality" "passing over into actual-

These two creation stories might also be illustrated as follows.

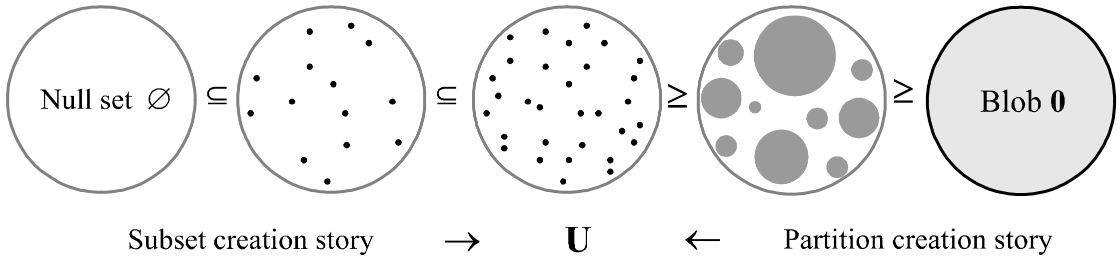


Figure 6: Two ways to create a universe U

One might think of the universe U (in the middle of the above picture) as the macroscopic world of fully definite entities that we ordinarily experience. Common sense and classical physics assumes, as it were, the subset creation story on the left. But *a priori,* it could just as well have been the dual story, the partition creation story pictured on the right, that leads to the *same* macro-picture U. And, as we will see, that is indeed the message of quantum mechanics.

5 The Lifting Program

5.1 From sets to vector spaces

We have so far outlined the mathematics of set partitions such as the representation of an indefinite element as a (non-singleton) block in a partition and carving out the fully distinct eigen-elements by making more distinctions, e.g., joining together the distinctions of different partitions (on the same universe). The lifting program lifts these set-based concepts to the much richer environment of vector spaces, particularly over the complex numbers.

Why vector spaces? Dirac [12] noted that the notion of superposition was basic to and characteristic of quantum mechanics. At the level of sets, there is only a very simple and austere notion of "superposition," namely collecting together definite eigen-elements into one subset interpreted as one indefinite element (indistinct between the "superposed" eigen-elements). In a vector space, superposition is represented by a weighted vector sum with weights drawn from the base field. Thus the lifting of set concepts to vector spaces gives a much richer version of partition mathematics, and, as we will see, the lifting gives the mathematics of quantum mechanics.

The lifting program is a conceptual "algorithm"[19] with the guiding:

Basis Principle:

Apply the set concept to a basis set and then generate the lifted vector space concept.[20]

ity by means of the form" should be seen as the actual indefinite "passing over into" the actual definite by being objectively in-formed through the making of distinctions.

[19] There are some choices involved so it is not an algorithm in the strict sense.

[20] Intuitions can be guided by the linearization map which takes a set U to the (free) vector space $\mathbb{C}^U$ where u lifts to the basis vector $\delta_u = \chi_{\{u\}} : U \to \mathbb{C}$. But some choices are involved in the lifting program. For instance, the set attribute $f : U \to \mathbb{R}$ could be taken as defining the functional $\mathbb{C}^U \to \mathbb{C}$ that takes δ_u to $f(u)$ or the operator $\mathbb{C}^U \to \mathbb{C}^U$ that takes δ_u to $f(u)\,\delta_u$. We will see that the latter is the right choice.

For instance, what is the vector space lift of the set concept of cardinality? We apply the set concept of cardinality to a basis set of a vector space where it yields the notion of *dimension* of the vector space (after checking that all bases have equal cardinality). Thus the lift of set-cardinality is not the cardinality of a vector space but its dimension.[21] Thus the null set $\emptyset$ with cardinality 0 lifts to the trivial zero vector space with dimension 0.

It is often convenient to refer to a set concept in terms of its lifted vector space concept. This will be done by using the name of the vector space concept enclosed in scare quotes, e.g., the cardinality of a set is its "dimension."

5.2 Lifting set partitions

To lift the mathematics of set partitions to vector spaces, the first question is the lift of a set partition. In the category of sets, the direct sum is the disjoint union, and the union of the blocks in a partition is a disjoint union. Hence a set partition is a direct sum decomposition of the universe set, so one might expect the corresponding vector space concept to be a direct sum decomposition of the space (where "direct sum" is defined in the category of vector spaces over some base field). That answer is immediately obtained by applying the set concept of a partition to a basis set and then seeing what it generates. Each block B of the set partition of a basis set generates a subspace $W_B \subseteq V$, and the subspaces together form a *direct sum decomposition*: $V = \sum_B \oplus W_B$. Thus the proper lifted notion of a partition for a vector space is *not* a set partition of the space as defined by a subspace $W \subseteq V$ where $v \sim v'$ if $v - v' \in W$,[22] but is a direct sum decomposition of the vector space.[23]

5.3 Lifting partition joins

The main partition operation that we need to lift to vector spaces is the join operation. Two set partitions cannot be joined unless they are *compatible* in the sense of being defined on the same universe set. This notion of compatibility lifts to vector spaces, via the basis principle, by defining two vector space partitions $\omega = \{W_\lambda\}$ and $\xi = \{X_\mu\}$ on V as being *compatible* if there is a basis set for V so that the two vector space partitions arise from two set partitions of that common basis set.

If two set partitions $\pi = \{B\}$ and $\sigma = \{C\}$ are compatible, then their *join* $\pi \vee \sigma$ is defined as the set partition whose blocks are the non-empty intersections $B \cap C$. Similarly the lifted concept is that if two vector space partitions $\omega = \{W_\lambda\}$ and $\xi = \{X_\mu\}$ are compatible, then their *join* $\omega \vee \xi$ is defined as the vector space partition whose subspaces are the non-zero intersections $W_\lambda \cap X_\mu$. And by the definition of compatibility, we could generate the subspaces of the join $\omega \vee \xi$ by the blocks in the join of the two set partitions of the common basis set.

[21] In QM, the extension of concepts on finite dimensional Hilbert space to infinite dimensional ones is well-known. Since our expository purpose is conceptual rather than mathematical, we will stick to finite dimensional spaces.

[22] The usual ad hoc quantum logic approach to define a 'propositional' logic for QM focused on the question of whether or not a vector was in a subspace, which in turn led to a misplaced focus on the set equivalence relations defined by the subspaces, equivalence relations that have a special property of being *commuting* [21]. If "quantum logic" is to be the logic that is to QM as Boolean subset logic is to classical mechanics, then that is partition logic, particularly as developed in "quantum mechanics" over $\mathbb{Z}_2$ (see below).

[23] Hermann Weyl calls a partition a "grating" or "sieve," and then correctly considers both set partitions and vector space partitions (direct sum decompositions) as the respective types of gratings. [44, pp. 255-257]

5.4 Lifting attributes

A set partition might be seen as an abstract rendition of the inverse image partition $\{f^{-1}(r)\}$ defined by some concrete attribute $f : U \to \mathbb{R}$ on U. What is the lift of an attribute? At first glance, the basis principle would seem to imply: define a set attribute on a basis set (with values in the base field) and then linearly generate a functional from the vector space to the base field. But a functional does not define a vector space partition; it only defines the set partition of the vector space compatible with the vector space operations that is determined by the kernel of the functional. Hence we need to try a more careful application of the basis principle.

It is helpful to first give a suggestive reformulation of a set attribute $f : U \to \mathbb{R}$. If f is constant on a subset $S \subseteq U$ with a value r, then we might symbolize this as:

$$f \upharpoonright S = rS$$

and suggestively call S an "eigenvector" and r an "eigenvalue." For any "eigenvalue" r, define $f^{-1}(r) =$ "eigenspace of r" as the union of all the "eigenvectors" with that "eigenvalue." Since the "eigenspaces" span the set U, the attribute $f : U \to \mathbb{R}$ can be represented by:

$$f = \sum_r r\chi_{f^{-1}(r)}$$

"Spectral decomposition" of set attribute $f : U \to \mathbb{R}$

[where $\chi_{f^{-1}(r)}$ is the characteristic function for the "eigenspace" $f^{-1}(r)$]. Thus a set attribute determines a set partition and has a constant value on the blocks of the set partition, so by the basis principle, that lifts to a vector space concept that determines a vector space partition and has a constant value on the blocks of the vector space partition.

The suggestive terminology gives the lift. The lift of $f \upharpoonright S = rS$ is the eigenvector equation $Lv = \lambda v$ where L is a linear operator on V. The lift of r is the eigenvalue λ and the lift of an S such that $f \upharpoonright S = rS$ is an eigenvector v such that $Lv = lv$. The lift of an "eigenspace" $f^{-1}(r)$ is the eigenspace W_λ of an eigenvalue λ. The lift of the simplest attributes, which are the characteristic functions $\chi_{f^{-1}(r)}$, are the projection operators P_λ that project to the eigenspaces W_λ. The characteristic property of the characteristic functions $\chi : U \to \mathbb{R}$ is that they are idempotent in the sense that $\chi(u)\chi(u) = \chi(u)$ for all $u \in U$, and the lifted characteristic property of the projection operators $P : V \to V$ is that they are idempotent in the sense that $P^2 : V \to V \to V = P : V \to V$. Finally, the "spectral decomposition" of a set attribute lifts to the spectral decomposition of a *vector space attribute*:

$$f = \sum_r r\chi_{f^{-1}(r)} \text{ lifts to } L = \sum_\lambda \lambda P_\lambda.$$

Lift of a set attribute to a vector space attribute

Thus a vector space attribute is just a linear operator whose eigenspaces span the whole space which is called a *diagonalizable linear operator* [26]. Then we see that the proper lift of a set attribute using the basis principle does indeed define a vector space partition, namely that of the eigenspaces of a diagonalizable linear operator, and that the values of the attribute are constant on the blocks of the vector space partition–as desired. To keep the eigenvalues of the linear operator real, quantum mechanics restricts the vector space attributes to *Hermitian* (or *self-adjoint*) linear operators, which represent *observables*, on a Hilbert space.

Hermann Weyl is one of the few quantum physicists who, in effect, outlined the lifting program by first considering an attribute on a set, which defined the set partition or "grating" [44, p. 255] of elements with the same attribute-value. Then he moved to the quantum case where the set or "aggregate of n states has to be replaced by an n-dimensional Euclidean vector space" [44, p. 256] (note the lift from sets to vector spaces using the basis principle). The appropriate notion of a partition or "grating" is a "splitting of the total vector space into mutually orthogonal subspaces" so that "each vector $\overrightarrow{x}$ splits into r component vectors lying in the several subspaces" [44, p. 256], i.e., a direct sum decomposition of the space, where the subspaces are the eigenspaces of an observable operator.

Lifting Program	Set concept	Vector space concept
Eigenvalues	r s.t. $f\restriction S = rS$ for some S	λ s.t. $Lv = \lambda v$ for some v
Eigenvectors	S s.t. $f\restriction S = rS$ for some r	v s.t. $Lv = \lambda v$ for some λ
Eigenspaces	$\{S: f\restriction S = rS\} = \wp(f^{-1}(r))$	$\{v: Lv = \lambda v\} = W_\lambda$
Partition	Set partition of "eigenspaces" $f^{-1}(r)$	Vector space partition of eigenspaces W_λ
Characteristic functions	$\chi_S: U \rightarrow \{0,1\}$ for subsets S like $f^{-1}(r)$	Projection operators for subspaces like $W_\lambda = P_\lambda(V)$
Spectral decomposition	Set attribute $f: U \rightarrow \mathbb{R}$: $f = \Sigma_r r\chi_{f^{-1}(r)}$	Hermitian linear operator: $L = \Sigma_\lambda \lambda P_\lambda$

Figure 7: Set attributes lift to linear operators

One of the mysteries of quantum mechanics is that the set attributes such as position or momentum on the phase spaces of classical physics become linear operators on the state spaces of QM. The lifting program "explains" that mystery.

5.5 Lifting compatible attributes

Since two set attributes $f: U \rightarrow \mathbb{R}$ and $g: U' \rightarrow \mathbb{R}$ define two inverse image partitions $\{f^{-1}(r)\}$ and $\{g^{-1}(s)\}$ on their domains, we need to extend the concept of compatible partitions to the attributes that define the partitions. That is, two attributes $f: U \rightarrow \mathbb{R}$ and $g: U' \rightarrow \mathbb{R}$ are *compatible* if they have the same domain $U = U'$.[24] We have previously lifted the notion of compatible set partitions to compatible vector space partitions. Since real-valued set attributes lift to Hermitian linear operators, the notion of compatible set attributes just defined would lift to two linear operators being *compatible* if their eigenspace partitions are compatible. It is a standard fact of QM math (e.g., [27, pp. 102-3] or [26, p. 177]) that two (Hermitian) linear operators $L, M: V \rightarrow V$ are compatible if and only if they commute, $LM = ML$. Hence the *commutativity* of linear operators

[24] This simplified definition is justified by the later treatment of compatible attributes in the context of "quantum mechanics" on sets.

is the lift of the compatibility (i.e., defined on the same set) of set attributes. Thus the join of two eigenspace partitions is defined iff the operators commute, i.e., as Weyl put it: "Thus combination [join] of two gratings [vector space partitions] presupposes commutability...". [44, p. 257]

Given two compatible set attributes $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$, the join of their "eigenspace" partitions has as blocks the non-empty intersections $f^{-1}(r) \cap g^{-1}(s)$. Each block in the join of the "eigenspace" partitions could be characterized by the ordered pair of "eigenvalues" (r, s). An "eigenvector" of f, $S \subseteq f^{-1}(r)$, and of g, $S \subseteq g^{-1}(s)$, would be a "simultaneous eigenvector": $S \subseteq f^{-1}(r) \cap g^{-1}(s)$.

In the lifted case, two commuting Hermitian linear operator L and M have compatible eigenspace partitions $W_L = \{W_\lambda\}$ (for the eigenvalues λ of L) and $W_M = \{W_\mu\}$ (for the eigenvalues μ of M). The blocks in the join $W_L \vee W_M$ of the two compatible eigenspace partitions are the non-zero subspaces $\{W_\lambda \cap W_\mu\}$ which can be characterized by the ordered pairs of eigenvalues (λ, μ). The nonzero vectors $v \in W_\lambda \cap W_\mu$ are *simultaneous eigenvectors* for the two commuting operators, and there is a basis for the space consisting of simultaneous eigenvectors.[25]

A set of compatible set attributes is said to be *complete* if the join of their partitions is discrete, i.e., the blocks have cardinality 1. Each element of U is then characterized by the ordered n-tuple $(r, ..., s)$ of attribute values.

In the lifted case, a set of commuting linear operators is said to be *complete* if the join of their eigenspace partitions is nondegenerate, i.e., the blocks have dimension 1. The eigenvectors that generate those one-dimensional blocks of the join are characterized by the ordered n-tuples $(\lambda, ..., \mu)$ of eigenvalues so the eigenvectors are usually denoted as the eigenkets $|\lambda, ..., \mu\rangle$ in the Dirac notation. These *Complete Sets of Commuting Operators* are Dirac's CSCOs [12].

5.6 Summary of lifting program

The lifting program so far is summarized in the following table.

[25] One must be careful not to assume that the simultaneous eigenvectors are the eigenvectors for the operator $LM = ML$ due to the problem of degeneracy.

Lifting Summary	Set concept	Vector space concept
Partition	Direct sum decomposition $\pi = \{B\}$ of U: $U = \uplus B$	Direct sum decomposition $\{W_i\}$ of V: $V = \sum \oplus W_i$
Real-valued Attribute	Function $f:U \to \mathbb{R}$	Hermitian operator $L: V \to V$
Partition of attribute	Inverse-image partition $\{f^{-1}(r)\}$ for $f:U \to \mathbb{R}$	Eigenspace partition $W_L = \{W_\lambda\}$ for $L: V \to V$
Compatible partitions	Partitions π,σ on same set U	Vector space partitions $\{W_i\}$ and $\{X_j\}$ with common basis
Compatible attributes	Attributes $f,g:U \to \mathbb{R}$ defined on same set U	Commuting operators LM = ML, i.e., common basis of simultaneous eigenvectors.
Join of compatible attribute partitions	$f^{-1} \vee g^{-1} = \{f^{-1}(r) \cap g^{-1}(s)\}$ for $f,g:U \to \mathbb{R}$	$W_L \vee W_M = \{W_\lambda \cap W_\mu\}$ for LM = ML
CSCO	Singleton blocks of $\vee f_i^{-1}$ for compatible attributes $\{f_i^{-1}\}$	One-dim. blocks of $\vee W_{L_i}$ for commuting operators $\{L_i\}$

Figure 8: Summary of Lifting Program

6 The Delifting Program: "Quantum mechanics" on sets

6.1 Probabilities in "quantum mechanics" on sets

The lifting program establishes a ladder between concepts and operations for sets and those for vector spaces. We have so far started with set concepts, like the concept of a set partition, and then climbed the ladder to the corresponding concept for vector spaces (direct sum decomposition). One can also climb down the ladder by delifting quantum mechanical concepts from vector spaces to sets. By delifting QM concepts to sets, we can develop a model called *"quantum mechanics" over* $\mathbb{Z}_2$–which distills out the metrical aspects of QM and shows the logical structure of QM in the pedagogically simple and understandable context of sets.[26] If we use the broader notion of the "logic" of something as giving the conceptual structure stripped down to essentials, then "quantum mechanics" over $\mathbb{Z}_2$ is indeed the *logic of quantum mechanics* (in that non-propositional logic sense of logic).[27]

The set version of QM is said to be "over $\mathbb{Z}_2$" since the power set $\wp(U)$ is a vector space over $\mathbb{Z}_2 = \{0,1\}$ [integers mod(2)] where the subset addition $S+T$ is the *symmetric difference* of subsets, i.e., $S+T = S \cup T - S \cap T$ for $S,T \subseteq U$. Thus set concepts can be first translated into sets-as-vectors concepts for vector spaces over $\mathbb{Z}_2$ and then lifted to vector spaces over $\mathbb{C}$ (or

[26] Recall that a delifted vector space concept is indicated by the concept's name in scare quotes.

[27] Instead of finding the naturally occurring logic that fits QM (i.e., partition logic), the standard approach [2] to "quantum logic" has been to just *ad hocly* define it as the "propositional logic" for QM statements (a state vector being in or not in an eigenspace). Since this "quantum logic" was defined in terms of QM, it can bring no interpretative insight to QM.

vice-versa for delifting) essentially by lifting, *mutatus mutandis*, from $\mathbb{Z}_2$ to $\mathbb{C}$.

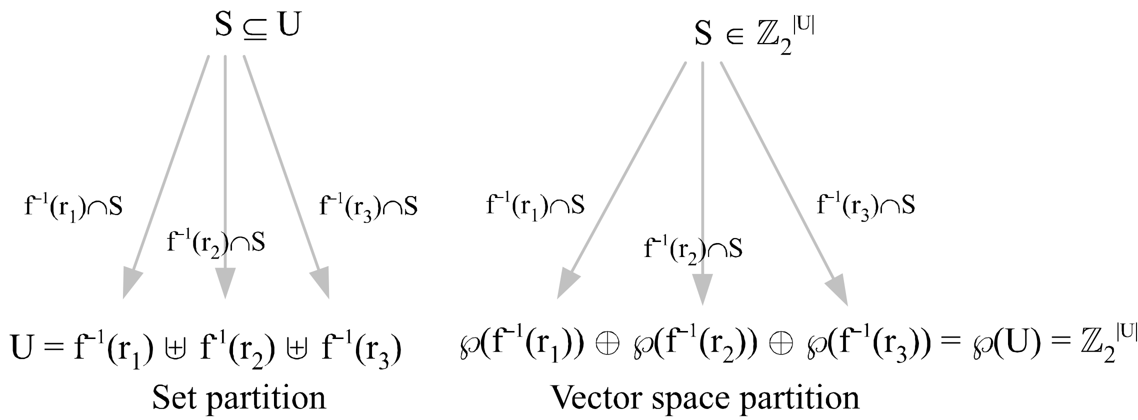


Figure 8a: Subsets $S \subseteq U$ linearized as vectors $S \in \mathbb{Z}_2^{|U|}$

A vector in $\mathbb{Z}_2^n$ is specified in the U-basis $\{\{u_1\},\{u_2\},...,\{u_n\}\}$ by its characteristic function $\chi_S : U \to \mathbb{Z}_2$, and a vector v in $\mathbb{C}^n$ is specified in terms of an orthonormal basis $\{|v_i\rangle\}$ by a function $\langle \_|v\rangle : \{v_i\} \to \mathbb{C}$ assigning a complex amplitude $\langle v_i|v\rangle$ to each basis vector. One of the key pieces of mathematical machinery in QM, namely the inner product, does not exist in vector spaces over finite fields but a basis-dependent "bracket" can be defined and a norm or absolute value can be defined to play a similar role in the probability algorithm of "quantum mechanics" on sets.

Seeing $\wp(U)$ as the vector space $\mathbb{Z}_2^{|U|}$ allows different bases in which the vectors can be expressed (as well as the basis-free notion of a vector as a ket, since only the bra is basis-dependent). Consider the simple case of $U = \{a,b,c\}$ where the U-basis is $\{a\}$, $\{b\}$, and $\{c\}$. But the three subsets $\{a,b\}$, $\{b,c\}$, and $\{a,b,c\}$ also form a basis since: $\{a,b\}+\{a,b,c\} = \{c\}$; $\{b,c\}+\{c\} = \{b\}$; and $\{a,b\}+\{b\} = \{a\}$. These new basis vectors could be considered as the basis-singletons in another equicardinal universe $U' = \{a',b',c'\}$ where $a' = \{a,b\}$, $b' = \{b,c\}$, and $c' = \{a,b,c\}$. In the following *ket table*, each row is a ket of $V = \mathbb{Z}_2^3$ expressed in the U-basis and the U'-basis.

$U = \{a,b,c\}$	$U' = \{a',b',c'\}$
$\{a,b,c\}$	$\{c'\}$
$\{a,b\}$	$\{a'\}$
$\{b,c\}$	$\{b'\}$
$\{a,c\}$	$\{a',b'\}$
$\{a\}$	$\{b',c'\}$
$\{b\}$	$\{a',b',c'\}$
$\{c\}$	$\{a',c'\}$
$\emptyset$	$\emptyset$

Vector space isomorphism (i.e., preserves +) $\mathbb{Z}_2^3 \cong \wp(U) \cong \wp(U')$: row = ket.

In a Hilbert space, the inner product is used to define the amplitudes $\langle v_i|v\rangle$ and the norm $\|v\| = \sqrt{\langle v|v\rangle}$, and the probability algorithm can be formulated using this norm. In a vector space over $\mathbb{Z}_2$, the Dirac notation can still be used but in a basis-dependent form (like matrices as opposed to operators) that defines a real-valued norm even though there is no inner product. The kets $|S\rangle$ for $S \in \wp(U)$ are basis-free but the corresponding bras are basis-dependent. For $u \in U$, the "*bra*" $\langle\{u\}|_U : \wp(U) \to \mathbb{R}$ is defined by the "*bracket*":

$$\langle\{u\}|_U S\rangle = \begin{cases} 1 \text{ if } u \in S \\ 0 \text{ if } u \notin S \end{cases} = \chi_S(u)$$

Then $\langle\{u'\}|_U \{u\}\rangle = \chi_{\{u\}}(u') = \chi_{\{u'\}}(u) = \delta_{u'u}$ is the delift of $\langle v_i|v_j\rangle = \delta_{ij}$. Assuming a finite U, the "bracket" linearly extends to the more general basis-dependent form (where $|S|$ is the cardinality of S):

$$\langle T|_U S\rangle = |T \cap S| \text{ for } T, S \subseteq U.^{28}$$

This delifting of the Dirac bracket is obtained by the basis principle in reverse. Consider an orthonormal basis set $\{|v_i\rangle\}$ in a finite dimensional Hilbert space. Given two subsets $T, S \subseteq \{|v_i\rangle\}$, consider the unnormalized superpositions $\psi_T = \sum_{|v_i\rangle \in T} |v_i\rangle$ and similarly for ψ_S. Then their inner product in the Hilbert space is $\langle\psi_T|\psi_S\rangle = |T \cap S|$, which "delifts" (running the basis principle in reverse to climb down the ladder) to $\langle T|_U S\rangle = |T \cap S|$ for subsets $T, S \subseteq U$ of the U-basis of $\mathbb{Z}_2^{|U|}$.

The basis-dependent "*ket-bra*" $|\{u\}\rangle\langle\{u\}|_U : \wp(U) \to \wp(U)$ is the "one-dimensional" projection operator $\{u\} \cap () : \wp(U) \to \wp(U)$ and the "ket-bra identity" holds as usual:

$$\sum_{u \in U} |\{u\}\rangle\langle\{u\}|_U = \sum_{u \in U} (\{u\} \cap ()) = I : \wp(U) \to \wp(U)$$

where the summation is the symmetric difference of sets in $\mathbb{Z}_2^n$. The overlap $\langle T|_U S\rangle$ can be resolved using the "ket-bra identity" in the same basis:

$$\langle T|_U S\rangle = \sum_u \langle T|_U \{u\}\rangle\langle\{u\}|_U S\rangle.$$

Then the (basis-dependent) U-*norm* $\|S\|_U : \wp(U) \to \mathbb{R}$ is defined, as usual, as the square root of the bracket:[29]

$$\|S\|_U = \sqrt{\langle S|_U S\rangle} = \sqrt{|S|}$$

for $S \in \wp(U)$ which is the delift of the basis-free norm $|\psi| = \sqrt{\langle\psi|\psi\rangle}$ (since the inner product does not depend on the basis). Note that a ket has to be expressed in the U-basis to apply the basis-dependent definition so in the above example, $\|\{a'\}\|_U = \sqrt{2}$ since $\{a'\} = \{a, b\}$ in the U-basis.

For a specific basis $\{|v_i\rangle\}$ and for any nonzero vector v in a finite dimensional vector space, $|v|^2 = \sum_i \langle v_i|v\rangle\langle v_i|v\rangle^*$ whose delifted version would be: $\|S\|_U^2 = \sum_{u \in U} \langle\{u\}|_U S\rangle^2$. Thus we also have:

$$\sum_i \frac{\langle v_i|v\rangle\langle v_i|v\rangle^*}{|v|^2} = 1 \text{ and } \sum_u \frac{\langle\{u\}|_U S\rangle^2}{\|S\|_U^2} = \sum_u \frac{|\{u\} \cap S|}{|S|} = 1$$

[28] Thus $\langle T|_U S\rangle = |T \cap S|$ takes values outside the base field of $\mathbb{Z}_2$ just like the Hamming distance function $|T + S|$ on vector spaces over $\mathbb{Z}_2$ in coding theory [32, p. 66] as applied to pairs of sets represented as binary strings.

[29] We use the double-line notation $\|S\|_U$ for the norm of a set to distinguish it from the single-line notation $|S|$ for the cardinality of a set, whereas the customary absolute value notation for the norm of a vector is $|v|$.

where $\frac{\langle v_i|v\rangle\langle v_i|v\rangle^*}{|v|^2}$ is a 'mysterious' quantum probability while $\frac{|\{u\}\cap S|}{|S|}$ is the unmysterious probability $\Pr(\{u\}|S)$ of getting u when sampling S (equiprobable elements of U). We previously saw that a subset $S \subseteq U$ as a block in a partition could be interpreted as a single indefinite element rather than a subset of definite elements. In like manner, we might interpret a subset of outcomes (an event) in a finite probability space as a single indefinite outcome where the conditional probability $\Pr(\{u\}|S)$ is the objective probability of a "U-measurement" of S yielding the definite outcome $\{u\}$.

An observable, i.e., a Hermitian operator, on a Hilbert space determines its home basis set of orthonormal eigenvectors. In a similar manner, an attribute $f : U \to \mathbb{R}$ defined on U has the U-basis as its "home basis set." Then given a Hermitian operator $L = \sum_\lambda \lambda P_\lambda$ and a U-attribute $f : U \to \mathbb{R}$, we have:

$$|v|^2 = \sum_\lambda |P_\lambda(v)|^2 \text{ and } \|S\|_U^2 = \sum_r \left\|f^{-1}(r) \cap S\right\|_U^2$$

where $f^{-1}(r) \cap S$ is the "projection operator" $f^{-1}(r) \cap ()$ applied to S, the delift of applying the projection operator P_λ to v.[30] This can also be written as:

$$\sum_\lambda \frac{|P_\lambda(v)|^2}{|v|^2} = 1 \text{ and } \sum_r \frac{\left\|f^{-1}(r)\cap S\right\|_U^2}{\|S\|_U^2} = \sum_r \frac{\left|f^{-1}(r)\cap S\right|}{|S|} = 1$$

where $\frac{|P_\lambda(v)|^2}{|v|^2}$ is the quantum probability of getting λ in an L-measurement of v while $\frac{\left|f^{-1}(r)\cap S\right|}{|S|}$ has the rather unmysterious interpretation of the probability $\Pr(r|S)$ of the random variable $f : U \to \mathbb{R}$ having the value r when sampling $S \subseteq U$. That is, the delift of the Born rule is not some weird "quantum" notion of probability on sets but the perfectly ordinary Laplace-Boole rule for the probability $\frac{\left|f^{-1}(r)\cap S\right|}{|S|}$, given $S \subseteq U$, of a random variable $f : U \to \mathbb{R}$ having the value r.

The usual completeness and orthogonality conditions on eigenspace partitions delift to the corresponding conditions in "QM" over $\mathbb{Z}_2$:

completeness: $\sum_\lambda P_\lambda = I : V \to V$ delifts to: $\sum_r f^{-1}(r) \cap () = I : \wp(U) \to \wp(U)$, and

orthogonality: for $\lambda \neq \lambda'$, $P_\lambda P_{\lambda'} = 0 : V \to V$ delifts to: for $r \neq r'$, $\left[f^{-1}(r) \cap ()\right]\left[f^{-1}(r') \cap ()\right] = \emptyset \cap () : \wp(U) \to \wp(U)$.

Under the set version of the objective indefiniteness interpretation, i.e., "quantum mechanics" on sets, the indefinite element S is being "measured" using the "observable" f where the probability $\Pr(r|S)$ of getting the "eigenvalue" r is $\frac{\left|f^{-1}(r)\cap S\right|}{|S|}$ and where the "damned quantum jump" goes from S to the "projected resultant state" which is the "eigenvector" $f^{-1}(r) \cap S$. That state represents a more-definite element that now has the definite f-value of r–so a second measurement would yield the same "eigenvalue" r and the same "eigenvector" $f^{-1}(r) \cap \left[f^{-1}(r) \cap S\right] = f^{-1}(r) \cap S$ (all as in the standard Dirac-von-Neumann treatment of measurement).

These delifts and more are summarized in the following table for a finite U and a finite dimensional Hilbert space V with $\{|v_i\rangle\}$ as any orthonormal basis.

[30] Since $\wp(U)$ is now interpreted as a vector space, it should be noted that the projection operator $S \cap () : \wp(U) \to \wp(U)$ is linear, i.e., $(S \cap S_1) + (S \cap S_2) = S \cap (S_1 + S_2)$. Indeed, this is the distributive law when $\wp(U)$ is interpreted as a Boolean ring.

Set Case: "QM"	Hilbert space case: QM
Projections $S\cap():\wp(U)\rightarrow\wp(U)$	$P:V\rightarrow V$
Spectral Decomp. $f\restriction()=\sum_r r\left(f^{-1}(r)\cap()\right)$	$L=\sum_\lambda \lambda P_\lambda$
Compl. $\sum_r f^{-1}(r)\cap()=I:\wp(U)\rightarrow\wp(U)$	$\sum_\lambda P_\lambda=I$
Orthog. $r\neq r'$, $\left[f^{-1}(r)\cap()\right]\left[f^{-1}(r')\cap()\right]=\emptyset\cap()$	$\lambda\neq\lambda'$, $P_\lambda P_{\lambda'}=0$
Brackets $\langle S\|_U T\rangle=\|S\cap T\|=$ overlap for $S,T\subseteq U$	$\langle\psi\|\varphi\rangle=$ "overlap" of ψ and φ
Ket-bra $\sum_{u\in U}\|\{u\}\rangle\langle\{u\}\|_U=\sum_{u\in U}(\{u\}\cap())=I$	$\sum_i \|v_i\rangle\langle v_i\|=I$
Resolution $\langle S\|_U T\rangle=\sum_u\langle S\|_U\{u\}\rangle\langle\{u\}\|_U T\rangle$	$\langle\psi\|\varphi\rangle=\sum_i\langle\psi\|v_i\rangle\langle v_i\|\varphi\rangle$
Norm $\\|S\\|_U=\sqrt{\langle S\|_U S\rangle}=\sqrt{\|S\|}$ where $S\subseteq U$	$\|\psi\|=\sqrt{\langle\psi\|\psi\rangle}$
Pythagoras $\\|S\\|_U^2=\sum_{u\in U}\langle\{u\}\|_U S\rangle^2=\|S\|$	$\|\psi\|^2=\sum_i\langle v_i\|\psi\rangle^*\langle v_i\|\psi\rangle$
Laplace $S\neq\emptyset$, $\sum_{u\in U}\frac{\langle\{u\}\|_U S\rangle^2}{\\|S\\|_U^2}=\sum_{u\in S}\frac{1}{\|S\|}=1$	$\|\psi\rangle\neq 0$, $\sum_i\frac{\langle v_i\|\psi\rangle^*\langle v_i\|\psi\rangle}{\|\psi\|^2}=1$
$\\|S\\|_U^2=\sum_r\left\\|f^{-1}(r)\cap S\right\\|_U^2=\sum_r\left\|f^{-1}(r)\cap S\right\|=\|S\|$	$\|\psi\|^2=\sum_\lambda\|P_\lambda(\psi)\|^2$
$S\neq\emptyset$, $\sum_r\frac{\left\\|f^{-1}(r)\cap S\right\\|_U^2}{\\|S\\|_U^2}=\sum_r\frac{\left\|f^{-1}(r)\cap S\right\|}{\|S\|}=1$	$\|\psi\rangle\neq 0$, $\sum_\lambda\frac{\|P_\lambda(\psi)\|^2}{\|\psi\|^2}=1$
Born Rule: $\Pr(r\|S)=\frac{\left\\|f^{-1}(r)\cap S\right\\|_U^2}{\\|S\\|_U^2}=\frac{\left\|f^{-1}(r)\cap S\right\|}{\|S\|}$	$\Pr(\lambda\|\psi)=\frac{\|P_\lambda(\psi)\|^2}{\|\psi\|^2}$
Average of attribute: $\langle f\rangle_S=\frac{\langle S\|_U f\restriction()\|S\rangle}{\langle S\|_U S\rangle}$	$\langle L\rangle_\psi=\frac{\langle\psi\|L\|\psi\rangle}{\langle\psi\|\psi\rangle}$.

Probability math for "QM" over $\mathbb{Z}_2$ and for QM

We have already seen how objective indefiniteness can be modeled in "QM" over $\mathbb{Z}_2$ by using the vector sum to collect together or superpose the eigen-states, e.g., $\{a,b\}=\{a\}+\{b\}$. When states are superposed, then there can be interference, e.g., $\{a,b\}+\{b,c\}=\{a,c\}$ (see Appendix 3 on the two-slit experiment). We have also seen how the objective nature of the states gives rise to probabilities by the Laplace-Boole rule, e.g., $\frac{|\{a,c\}|}{|\{a,b,c\}|}=\frac{2}{3}$, so we can make sense out of objective probabilities (e.g., when we relift to the quantum probabilities on the right side of the above table). In the later treatment of "measurement" in "QM" over $\mathbb{Z}_2$, we will see how an attribute-observable $f:U\rightarrow\mathbb{R}$ can be used to make distinctions (i.e., make a "measurement"), and how those distinctions are counted by logical entropy.

6.2 Relifting logical probability theory to quantum mechanics

More QM mathematics such as density matrices will be delifted below, but the above table is sufficient proof of concept to illustrate the results of delifting the mathematical machinery of quantum mechanics on (finite dimensional) Hilbert spaces to sets. The result was not some strange "quantum theory" of sets; it was ordinary logical finite probability theory that was slightly generalized by taking the event-subsets of the outcome-set universe U as superposition-vectors (indefinite states) in $\mathbb{Z}_2^{|U|}$ so that the consideration of other bases would bring in non-commutativity. The specific form in which this non-commutative logical finite probability theory was stated was obviously influenced by the delift from QM, e.g., the formulation of the overlap $|S\cap T|=\langle S|_U T\rangle$ as a basis-dependent bracket so that the set version of Dirac's bra-ket calculus can be developed. Since vector spaces over finite fields have no inner products, the only cost is taking the bracket's value outside the base field (like the Hamming distance function), making the bras basis-dependent, and postponing normalization to the calculation of probabilities.

There are several payoffs. One is that "QM" over $\mathbb{Z}_2$ gives the pure "logic" of QM with all the metrical aspects distilled out so it is the logic of quantum mechanics (where "logic" is taken in the old non-propositional sense of displaying the essentials of the matter).

But the largest payoff is obtained by relifting non-commutative logical finite probability theory, with partitions supplied by the attributes, to obtain the mathematical machinery of quantum mechanics (finite dimensional case). The relifting process boils down to the ascent in base fields:

$$\mathbb{Z}_2 \Rightarrow \mathbb{C}$$

Relifting logical probability theory to quantum mechanics

with the *mutatis mutandis* changes such as taking advantage of the inner product that becomes available when the base field $\mathbb{Z}_2$ is replaced by the complex numbers $\mathbb{C}$ to give a basis-free bracket $\langle\psi|\varphi\rangle$ with values in the base field, normalization along the way (rather than only in the probability algorithm), and replacing the square (as in $\langle\{u\}|_U S\rangle^2$) with the *absolute* square (as in $\langle v_i|\psi\rangle\langle v_i|\psi\rangle^*$). This ascent from subsets to vectors in Hilbert space uses the old idea of a complex vector as a complex-valued set. That is, given a basis $\{|\{u\}\rangle\}_{u\in U}$ for $\mathbb{Z}_2^{|U|}$, a ket-subset $|S\rangle$ expressed in that basis is given by a function χ_S that takes $|\{u\}\rangle$ to $\chi_S(u) = \langle\{u\}|_U S\rangle \in \mathbb{Z}_2$ so that $|S\rangle = \sum_u \langle\{u\}|_U S\rangle |\{u\}\rangle$. Similarly given a basis $\{v_i\}$ for a Hilbert space, a ket-vector $|v\rangle$ expressed in that basis is given by a "characteristic function" $\langle\_|v\rangle$ that takes v_i to $\langle v_i|v\rangle \in \mathbb{C}$ so that $|v\rangle = \sum_i \langle v_i|v\rangle |v_i\rangle$.

This shows that the mathematical machinery of quantum mechanical probabilities has nothing to do with empirical physics *per se*; it is simply the restatement of logical probability theory with attribute-partitions over the complex numbers $\mathbb{C}$ instead of over $\mathbb{Z}_2$. The empirical physics only comes in by identifying the appropriate Hermitian operators such as the energy (Hamiltonian), momentum, angular momentum, and position (concepts from classical physics, not logic) as well as the DeBroglie relations relating energy with frequency and momentum with wave-number in the "wave-mathematics" that comes in solely with the ascent to the complex numbers as the base field–there being no physical waves in QM (see below).

The thesis is that the mathematical framework of quantum mechanical probabilities *is* the result of lifting non-commutative logical probability theory to the complex numbers, i.e.,

QM probability mathematics = logical probability theory with attribute-partitions linearized over $\mathbb{C}$.

This mathematical thesis is independent of any physical interpretation of QM. It has always been known that there would be no mystery in, say, the "collapse of the wave packet" if it was only interpreted as a collapse in a purely subjective "cloud of ignorance" as when sampling a probability space. Given an event $S \subseteq U$, the sampling returns a specific outcome $u \in U$. In the objective indefiniteness interpretation of "QM" over $\mathbb{Z}_2$, a subset-event $S \subseteq U$ is interpreted as a state that is indefinite between the basis states $\{u\} \subseteq S$, so the sampling-measurement is interpreted as the distinction-making event where that objective information reduces the indefinite state S to some definite $\{u\} \subseteq S$ with the probability $\langle\{u\}|_U S\rangle^2 / \langle S|_U S\rangle^2 = 1/|S|$. A random variable $f : U \to \mathbb{R}$ on the outcome space becomes, under the objective indefiniteness interpretation, an attribute-observable that has a definite "eigenvalue" r on the states contained in the "eigenspace" $f^{-1}(r)$, so the sampling-measurement of the indefinite state S using the random-variable-observable returns the "eigenvalue" r with the objective probability

$$\Pr(r|S) = \frac{\left\|f^{-1}(r)\cap S\right\|^2}{\|S\|^2} = \frac{\left|f^{-1}(r)\cap S\right|}{|S|}$$

and projects the indefinite state S into the definite state $f^{-1}(r) \cap S$ (see more on measurement in the next section).

Many of the aspects of QM–that are controversial in that setting–survive in the delifting to "QM" over $\mathbb{Z}_2$ where much of the mystery is dissipated. For instance, there is little controversy in the bracket $\langle S|_U S\rangle$ being resolved as the sum of "absolute squares" $\langle S|_U S\rangle = \sum_u \langle S|_U \{u\}\rangle \langle \{u\}|_U S\rangle$ of the "amplitudes" $\langle \{u\}|_U S\rangle$ giving the norm squared $\langle S|_U S\rangle = \|S\|^2 = |S|$. As we will see below and in the appendices, such controversial and/or mysterious aspects such as Heisenberg's indeterminacy principle, the two-slit experiment, Bell's Theorem, and the measurement problem can all be appropriately rendered in "QM" over $\mathbb{Z}_2$ where some of the mystery can be dispelled.

Thus the delifting program can be seen as a type of canonical procedure to try to answer–within the whole program–some of the questions of quantum philosophy. But it should be noted that this procedure may simplify but will not dispel the fact that objective indefiniteness is not our usual common-sense definite-properties-all-the-way-down view of things. In that respect, we need to cobble together various metaphors and analogies to assist in visualization, e.g., the previous Guy Fawkes masks, the clear and complete photograph of a cloud, the hawk's "flights and perchings" view (versus the hound's view) of getting from A to B, or the two creation stories (based on going from bottom to top of the two dual lattices).

6.3 Measurement in "QM" on sets

6.3.1 Non-degenerate measurement

Certainly the notion of measurement is one of the most opaque notions of QM so let's consider a set version of (projective) measurement starting at some block (the "state") in a partition in a partition lattice. In the simple example illustrated below we start at the one block or "state" of the indiscrete partition or blob which is the completely indistinct element $\{a, b, c\}$. A measurement always uses some attribute that defines an inverse-image partition on $U = \{a, b, c\}$.[31] In the case at hand, there are "essentially" four possible attributes that could be used to "measure" the indefinite element $\{a, b, c\}$ (since there are four partitions that refine the blob).

For an example of a "nondegenerate measurement," consider any attribute $f : U \to \mathbb{R}$ which has the discrete partition as its inverse image, such as the ordinal number of a letter in the alphabet: $f(a) = 1$, $f(b) = 2$, and $f(c) = 3$. This attribute or "observable" has three "eigenvectors": $f \restriction \{a\} = 1\{a\}$, $f \restriction \{b\} = 2\{b\}$, and $f \restriction \{c\} = 3\{c\}$ with the corresponding "eigenvalues." The "eigenspaces" in the inverse image are also $\{a\}$, $\{b\}$, and $\{c\}$, the blocks in the discrete partition of U all of which have "dimension" (i.e., cardinality) one. Starting in the "state" $S = \{a, b, c\}$, a U-measurement with this observable would yield the "eigenvalue" r with the probability of $\Pr(r|S) = \frac{\left|f^{-1}(r)\cap S\right|}{|S|} = \frac{1}{3}$. A "projective measurement" makes distinctions in the measured "state" that are sufficient to induce the "quantum jump" or "projection" to the "eigenvector" associated with the observed "eigenvalue." If the observed "eigenvalue" was 3, then the "state" $\{a, b, c\}$ "projects" to $f^{-1}(3) \cap \{a, b, c\} = \{c\} \cap \{a, b, c\} = \{c\}$ as pictured below.

[31] Weyl refers to a partition as a "grating" or "sieve" and then notes that "Measurement means application of a sieve or grating." [44, p. 259]

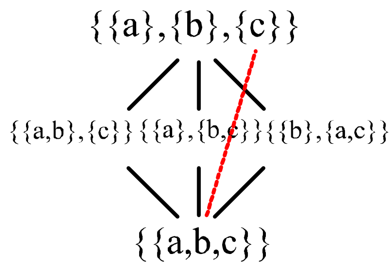


Figure 9: "Nondegenerate measurement"

It might be emphasized that this is an objective state reduction (or "collapse of the wave packet") from the single indefinite element $\{a, b, c\}$ to the single definite element $\{c\}$, not a subjective removal of ignorance as if the "state" had all along been $\{c\}$. For instance, Pascual Jordan in 1934 argued that:

> the electron is forced to a decision. We compel it to assume a definite position; previously, in general, it was neither here nor there; it had not yet made its decision for a definite position... . ... [W]e ourselves produce the results of the measurement. (quoted in [30, p. 161])

This might be illustrated using Weyl's notion of a partition as a "sieve or grating" [44, p. 259] that is applied in a measurement. We might think of a grating as a series of regular polygonal shapes that might be imposed on an indefinite blob of dough. In a measurement, the blob of dough falls through one of the polygonal holes with equal probability and then takes on that shape.

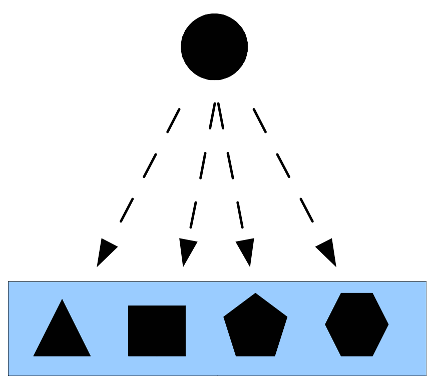


Figure 9.5: Measurement as randomly giving an indefinite blob of dough a regular polygonal shape.

6.3.2 Density matrices in "QM" on sets

It may be useful (for later purposes) to formulate this measurement using density matrices and logical entropy. Given a partition $\pi = \{B\}$ on $U = \{1, 2, ..., n\}$. If the points had the probabilities $p = \{p_1, ..., p_n\}$,the $n \times n$ "density matrix" $\rho(\pi)$ representing π has the entries:

$$\rho_{ij}(\pi) = \begin{cases} \sqrt{p_i p_j} \text{ if } (i,j) \in \operatorname{indit}(\pi) \\ 0 \text{ if } (i,j) \in \operatorname{dit}(\pi). \end{cases}$$

All the entries are real "amplitudes" whose squares are probabilities. Since the diagonal pairs (i,i) are always indits of a partition, the probabilities p_i are the diagonal entries. To foreshadow the quantum case, the non-zero off-diagonal entries $\sqrt{p_i p_j}$ indicate that i and j "cohere" together in a block of the partition. After interchanging some rows and the corresponding columns, the density matrix would be a block-diagonal matrix with the blocks corresponding to the blocks B of the partition π.

The *quantum logical entropy* of a density matrix ρ is: $h(\rho) = 1 - \operatorname{tr}\left[\rho^2\right]$, and the logical entropy of a set partition π with point probabilities p would be: $h(\rho(\pi)) = 1 - \operatorname{tr}\left[\rho(\pi)^2\right]$–which generalizes $h(p) = 1 - \sum_{i=1}^{n} p_i^2$ where π is the discrete partition on U. In the case at hand, all the points are considered equiprobable with $p_i = \frac{1}{n}$. Then a little calculation shows that:

$$h(\rho(\pi)) = 1 - \operatorname{tr}\left[\rho(\pi)^2\right] = 1 - \sum_{B\in\pi} p_B^2 = h(\pi)$$

where $p_B = \frac{|B|}{|U|} = \frac{|B|}{n}$.

A *pure-state* is given by a subset $S \subseteq U$ where $p_i = 0$ for all $i \notin S$ and $p_i > 0$ for $i \in S$. The state might be represented by a $n \times 1$ column vector $\left[\sqrt{p_1}, \ldots \sqrt{p_n}\right]^t$ of probability amplitudes so that its *pure-state density matrix* would be the $n \times n$ matrix obtained by multiplying the column vector times its transpose:

$$\rho(S) = \begin{bmatrix} \sqrt{p_1} \\ \vdots \\ \sqrt{p_n} \end{bmatrix} \begin{bmatrix} \sqrt{p_1} & \cdots & \sqrt{p_n} \end{bmatrix} = \begin{bmatrix} p_1 & \sqrt{p_1 p_2} & \cdots & \sqrt{p_1 p_n} \\ \sqrt{p_2 p_1} & p_2 & \cdots & \sqrt{p_2 p_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{p_n p_1} & \sqrt{p_n p_2} & \cdots & p_n \end{bmatrix}.$$

A pure-state density matrix

In "QM" over $\mathbb{Z}_2$, the points are equiprobable. For $U = \{a,b,c\}$, the "density matrix" (in the U-basis) of the state U is the constant pure-state density matrix taking $S = U$:

$$\rho(U) = \begin{bmatrix} \sqrt{1/3} \\ \sqrt{1/3} \\ \sqrt{1/3} \end{bmatrix} \begin{bmatrix} \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

The diagonal entries are the point probabilities. The off-diagonal (as well as diagonal) entries $\rho_{ij}(U)$ are the square roots $\sqrt{\frac{1}{3}\frac{1}{3}}$ of the probabilities of drawing the (i,j) pair (in two independent draws) if that pair is an indistinction of the partition, and otherwise the entry is 0. Since all pairs are indistinctions of the indiscrete partition $\mathbf{0} = \{U\}$ (i.e., cohere together in this indiscrete = "pure" state), all the off-diagonal elements of $\rho(U)$ are $\sqrt{\frac{1}{3}\frac{1}{3}} = \frac{1}{3}$. In general, $\operatorname{tr}\left[\rho(\pi)^2\right]$ will be the probability of drawing an indistinction of the partition, so $h(\rho(\pi)) = 1 - \operatorname{tr}\left[\rho(\pi)^2\right]$ is the probability of drawing a distinction of π.

Then,

$$h(\rho(U)) = 1 - \operatorname{tr}\left[\rho(U)^2\right] = 1 - \left(\tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3}\right) = 0$$

as we expect since there are no distinctions in the indiscrete partition (everything "coheres" together), and the interpretation of the logical entropy is the probability of drawing a distinction.

Given an attribute $f : U = \{1, ..., n\} \to \mathbb{R}$, the matrix representing this attribute in "QM" over $\mathbb{Z}_2$ is:

$$f = \begin{bmatrix} f(1) & 0 & \cdots & 0 \\ 0 & f(2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(n) \end{bmatrix}.$$

Given a subset $S \subseteq U$, the "density matrix" for that state has $p_i = \frac{1}{|S|}$ for $i \in S$ and $p_i = 0$ otherwise. With some column and row interchanges, the matrix would have a constant $|S| \times |S|$ block with the values $1/|S|$ and zeros elsewhere:

$$\rho(S) = \begin{bmatrix} \frac{1}{|S|} & \cdots & \frac{1}{|S|} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{|S|} & \cdots & \frac{1}{|S|} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then we have the result that the average value of an operator in a state given by a density matrix is the trace of the product:

$$\begin{aligned} \langle f \rangle_S = \operatorname{tr}[f\rho(S)] = \tfrac{1}{|S|} \textstyle\sum_{i \in S} f(i) = \tfrac{1}{|S|} \sum_{i \in U} f(i) \langle S|_U \{i\}\rangle \langle \{i\}|_U S\rangle \\ = \tfrac{1}{|S|} \langle S|_U f \restriction () \textstyle\sum_i |\{i\}\rangle \langle \{i\}|_U |S\rangle = \frac{\langle S|_U f \restriction () |S\rangle}{\langle S|_U S\rangle} \end{aligned}$$

where $f \restriction |\{i\}\rangle = f(i) |\{i\}\rangle$.

A real-valued "observable" is a set attribute $f : U \to \mathbb{R}$ which defines an inverse-image partition of "eigenspaces" $\left\{f^{-1}(r)\right\}$. Recall from the logic of partitions that the blocks of the join $\pi \vee \sigma$ of two partitions $\pi = \{B\}$ and $\sigma = \{C\}$ are the non-empty intersections $B \cap C$. This action of the join operation could be considered as a set of projection operators $\{B \cap ()\}_{B \in \pi}$ acting on the blocks $C \in \sigma$–or on a single subset $S \subseteq U$. The measurement according to the "observable" f is applied to some "pure state" $S \subseteq U$. The partition $f^{-1} = \left\{f^{-1}(r)\right\}$ acts as a set of projection operators $f^{-1} \vee () = \left\{f^{-1}(r) \cap ()\right\}$ on the "pure-state" S to partition it into the parts $f^{-1} \vee (S) = \left\{f^{-1}(r) \cap S\right\}$.

Given the density matrix of the "pure state" S, the density matrix resulting from the measurement is the "mixed state" density matrix for the results of the join operation $f^{-1} \vee (S)$. The density matrix of the action of the join $f^{-1} \vee (S)$ allows us to state the general result of a measurement without assuming a particular outcome. Note that representing the action of the measurement of the f-attribute applied to the state S as the projections $f^{-1}(r) \cap S$ is just the set version of the usual notion of projective measurement where $f^{-1}(r)\cap()$ plays the role of the projection operator P_λ to the eigenspace of an observable F for the eigenvalue λ.

In the philosophical literature on quantum mechanics, there is the "measurement problem" about the laws of motion of the quantum state in a measurement. We can now see this problem in the simple context of "QM" over $\mathbb{Z}_2$. The "law of motion" that transforms the initial-state S into the post-measurement "mixed state" $f^{-1}\vee(S)$ is the *join-action* of the "eigenspace" partition of the measurement "observable" $f^{-1}\vee() = \left\{f^{-1}(r)\cap()\right\}$ on the initial-state S. Thus the pre-measurement density matrix $\rho(S)$ is transformed by the measurement into the post-measurement density matrix $\hat{\rho}(S) = \rho\left(f^{-1}\vee(S)\right)$. This is simply the "QM" over $\mathbb{Z}_2$ version of the usual (von Neumann) notion of a projective measurement.[32]

A subset S represents an objectively indefinite state (e.g., of a particle) and it is transformed by the measurement into one of a number of more definite states $f^{-1}(r)\cap S$ (e.g., a particle with the definite f-attribute value of r) with the probabilities $\Pr(r|S) = \left|f^{-1}(r)\cap S\right|/|S|$. Any (non-trivial) measurement will refine S into smaller blocks which means making distinctions. Pairs of eigen-elements $u, u' \in U$ that "cohered" together by both being elements of S [two elements in the same subset are an indistinction] may be "decohered" into distinctions because they are now in separate blocks $\left\{f^{-1}(r_1)\cap S, f^{-1}(r_2)\cap S, ...\right\}$ of the sub-divided S. Hence the distinction-making changes made in the measurement process cannot be described as a process that preserves distinctions and indistinctions–where the latter process is, as we will see later, the only way an *isolated* system (i.e., no interactions with a distinction-making system) can change.[33] These two types of processes are the "QM" over $\mathbb{Z}_2$ versions of von Neumann's type 1 (distinction-making) and type 2 (distinction-preserving) processes [41, p. 351].

In the above example of a non-degenerate measurement, we assumed a particular outcome of the eigenvalue 3 and the eigenstate $\{c\}$. But the density matrix allows the more general formulation that the measurement turns the "pure state" $\rho(U)$ into a "mixed state" $\hat{\rho}(U) = \rho\left(\left\{f^{-1}(r)\right\}\vee(U)\right)$ without specifying the particular outcome. Since the measurement was non-degenerate, i.e., $\left\{f^{-1}(r)\right\} = \mathbf{1}$, the "mixed state density matrix" is diagonal:

$$\rho(U) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \Longrightarrow \hat{\rho}(U) = \rho(\mathbf{1}\vee(U)) = \rho(\mathbf{1}) = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Non-degenerate measurement: join-action of $U \Longrightarrow \left\{f^{-1}(r)\right\}\vee(U)$

which indicates that the measurement could have produced $\{a\}$, $\{b\}$, or $\{c\}$ each with probability $\frac{1}{3}$. Each of the off-diagonal terms was "decohered" by the non-degenerate measurement so the post-measurement "amplitude" of (i, j) still "cohering" is 0.[34]

The general result is that the logical entropy increase resulting from a measurement is the sum of the new distinction probabilities created by the join, which is the sum of the squared amplitudes of the off-diagonal indistinction amplitudes in the density matrix that were zeroed or "decohered" by the measurement. In this case, the six off-diagonal amplitudes of $\frac{1}{3}$ were all zeroed so the change in logical entropy is: $6\times\left(\frac{1}{3}\right)^2 = \frac{6}{9} = \frac{2}{3}$. Since the initial logical entropy was 0, that is also just

[32] See Appendix 5 on the "measurement problem" in "QM" on sets for further elaboration.

[33] See the later section on "indistinctness-preserving evolution."

[34] Note that this set-version of "decoherence" means actual reduction of state, not a "for all practical purposes" or FAPP [1] reduction.

the logical entropy of the discrete partition: $h(\mathbf{1}) = 1 - \sum_{i=1}^{3}\left(\frac{1}{3}\right)^2 = 1 - \frac{3}{9} = \frac{2}{3}$, which is also the logical entropy of the density matrix for that partition: $h(\rho(\mathbf{1})) = 1 - \operatorname{tr}\left[\rho(\mathbf{1})^2\right]$.

6.3.3 Degenerate measurements in "QM" on sets

For an example of a "degenerate measurement," we choose an attribute with a non-discrete inverse-image partition such as $\pi = \{\{a\},\{b,c\}\}$, which could, for instance, just be the characteristic function $\chi_{\{b,c\}}$ with the two "eigenspaces" $\{a\}$ and $\{b,c\}$ and the two "eigenvalues" 0 and 1 respectively. Since one of the two "eigenspaces" is not a singleton of an eigen-element, the "eigenvalue" of 1 is a set version of a "degenerate eigenvalue." This attribute $\chi_{\{b,c\}}$ has four "eigenvectors": $\chi_{\{b,c\}} \restriction \{b,c\} = 1\{b,c\}$, $\chi_{\{b,c\}} \restriction \{b\} = 1\{b\}$, $\chi_{\{b,c\}} \restriction \{c\} = 1\{c\}$, and $\chi_{\{b,c\}} \restriction \{a\} = 0\{a\}$.

The "measuring apparatus" makes distinctions by "joining" the "observable" partition $\chi_{\{b,c\}}^{-1} = \left\{\chi_{\{b,c\}}^{-1}(1), \chi_{\{b,c\}}^{-1}(0)\right\}$ with the "pure state" which is the single block representing the indefinite element $S = U = \{a,b,c\}$. A measurement apparatus of that "observable" returns one of "eigenvalues" with certain probabilities:

$$\Pr(0|S) = \frac{|\{a\}\cap\{a,b,c\}|}{|\{a,b,c\}|} = \frac{1}{3} \text{ and } \Pr(1|S) = \frac{|\{b,c\}\cap\{a,b,c\}|}{|\{a,b,c\}|} = \frac{2}{3}.$$

Suppose it returns the "eigenvalue" 1. Then the indefinite element $\{a,b,c\}$ "jumps" to the "projection" $\chi_{\{b,c\}}^{-1}(1) \cap \{a,b,c\} = \{b,c\}$ of the "state" $\{a,b,c\}$ to that "eigenvector" [9, p. 221].

In the density matrix treatment of this degenerate measurement, the result is the mixed state with the density matrix corresponding to the partition $\chi_{\{b,c\}}^{-1} \vee (U) = \chi_{\{b,c\}}^{-1} = \{\{a\},\{b,c\}\}$ which is:

$$\rho(U) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \Longrightarrow \hat{\rho}(U) = \rho\left(\chi_{\{b,c\}}^{-1} \vee U\right) = \rho\left(\chi_{\{b,c\}}^{-1}\right) = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

First degenerate measurement: join-action of $U \Longrightarrow \chi_{\{b,c\}}^{-1} \vee (U)$.

The logical entropy increase is the sum of off-diagonal terms squared that were zeroed or "decohered" by the measurement which in this case is: $4 \times \left(\frac{1}{3}\right)^2 = \frac{4}{9}$. This is also obtained as:

$$h(\hat{\rho}(U)) = 1 - \operatorname{tr}\left[\rho\left(\chi_{\{b,c\}}^{-1}\right)^2\right] = 1 - \operatorname{tr}\begin{bmatrix} \frac{1}{9} & 0 & 0 \\ 0 & \frac{2}{9} & \frac{2}{9} \\ 0 & \frac{2}{9} & \frac{2}{9} \end{bmatrix} = 1 - \frac{5}{9} = \frac{4}{9}.$$

In simpler terms, the new dits created by the measurement that took the indiscrete partition $\mathbf{0} = \{\{a,b,c\}\}$ to the partition $\chi_{\{b,c\}}^{-1} = \{\{a\},\{b,c\}\}$ are $\{(a,b),(a,c),(b,a),(c,a)\}$ (which correspond to the four zeroed off-diagonal terms) so the change in the normalized count of the dit sets is:

$$\frac{|\{(a,b),(a,c),(b,a),(c,a)\}|}{|U\times U|} = \frac{4}{9}.$$

Since this is a "degenerate" result (i.e., the "eigenspaces" don't all have "dimension" one), another measurement is needed to make more distinctions. Measurements by attributes that give either of the other two partitions, $\{\{a,b\},\{c\}\}$ or $\{\{b\},\{a,c\}\}$, suffice to distinguish $\{b,c\}$ into

$\{b\}$ or $\{c\}$, so either attribute together with the attribute $\chi_{\{b,c\}}$ would form a *complete set of compatible attributes* (i.e., the set version of a CSCO). The join of the two attributes' partitions gives the discrete partition. Taking the other attribute as $\chi_{\{a,b\}}$, the join of the two attributes' "eigenspace" partitions is discrete:

$$\chi_{\{b,c\}}^{-1} \vee \chi_{\{a,b\}}^{-1} = \{\{a\},\{b,c\}\} \vee \{\{a,b\},\{c\}\} = \{\{a\},\{b\},\{c\}\} = \mathbf{1}.$$

Hence all the "eigenstate" singletons can be characterized by the ordered pairs of the "eigenvalues" of these two "observables": $\{a\} = |0,1\rangle$, $\{b\} = |1,1\rangle$, and $\{c\} = |1,0\rangle$ (using Dirac's ket-notation to give the ordered pairs).

The second "projective measurement" of the indefinite "superposition" element $\{b,c\}$ using the attribute $\chi_{\{a,b\}}$ with the "eigenspace" partition $\chi_{\{a,b\}}^{-1} = \{\{a,b\},\{c\}\}$ would induce a jump to either $\{b\}$ or $\{c\}$ with the probabilities:

$$\Pr(1|\{b,c\}) = \frac{|\{a,b\}\cap\{b,c\}|}{|\{b,c\}|} = \frac{1}{2} \text{ and } \Pr(0|\{b,c\}) = \frac{|\{c\}\cap\{b,c\}|}{|\{b,c\}|} = \frac{1}{2}.$$

If the measured "eigenvalue" is 0, then the "state" $\{b,c\}$ "projects" to $\chi_{\{a,b\}}^{-1}(0) \cap \{b,c\} = \{c\}$ as pictured below.

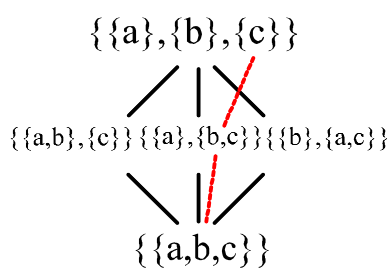


Figure 10: "Degenerate measurement"

The two "projective measurements" of $\{a,b,c\}$ using the complete set of compatible (both defined on U) attributes $\chi_{\{b,c\}}$ and $\chi_{\{a,b\}}$ produced the respective "eigenvalues" 1 and 0, and the resulting "eigenstate" was characterized by the "eigenket" $|1,0\rangle = \{c\}$.

Starting with the "mixed state" $\chi_{\{b,c\}}^{-1}$, the second measurement makes the following changes in the density matrices:

$$\rho\left(\chi_{\{b,c\}}^{-1}\right) = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \Rightarrow \rho\left(\chi_{\{a,b\}}^{-1} \vee \left(\chi_{\{b,c\}}^{-1}\right)\right) = \rho(\mathbf{1}) = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

Second degenerate measurement: join-action of $\chi_{\{b,c\}}^{-1} \Longrightarrow \chi_{\{a,b\}}^{-1} \vee \left(\chi_{\{b,c\}}^{-1}\right)$.

The off-diagonal terms zeroed by that measurement were the (b,c) and (c,b) terms (also the new dits) so the increase in logical entropy is $2 \times \left(\frac{1}{3}\right)^2 = \frac{2}{9}$, which can also be computed as:

$$h\left(\rho\left(\chi^{-1}_{\{a,b\}}\vee\left(\chi^{-1}_{\{b,c\}}\right)\right)\right)-h\left(\rho\left(\chi^{-1}_{\{b,c\}}\right)\right)$$
$$=\left\{1-\operatorname{tr}\begin{bmatrix}\frac{1}{9}&0&0\\0&\frac{1}{9}&0\\0&0&\frac{1}{9}\end{bmatrix}\right\}-\left\{1-\operatorname{tr}\begin{bmatrix}\frac{1}{9}&0&0\\0&\frac{2}{9}&\frac{2}{9}\\0&\frac{2}{9}&\frac{2}{9}\end{bmatrix}\right\}$$
$$=\left(1-\tfrac{3}{9}\right)-\left(1-\tfrac{5}{9}\right)=\tfrac{2}{9}.$$

The normalized dit count in going from the indiscrete to the discrete partition is same regardless of going all in one non-degenerate measurement or in two or more steps: $\frac{2}{3}=\frac{4}{9}+\frac{2}{9}$.

In this manner, the model of "quantum mechanics" over $\mathbb{Z}_2$ provides a set version of:

- a "nondegenerate measurement" by an "observable,"
- "degenerate measurements" by "compatible observables,"
- "projections" associated with "eigenvalues" that "project" to "eigenvectors,"
- characterizations of "eigenvectors" by "eigenkets" of "eigenvalues,"
- differentiation of "type 1" = distinction-changing processes from "type 2" = distinction-preserving processes,
- join-action of "measurement" given by application of projection operators $\left\{f^{-1}(r)\cap()\right\}$ to the given state S,
- "density matrix" treatments of the "measurements,"

$$\rho\,(\text{state})\Longrightarrow\rho\left(\text{"observable" partition } f^{-1}\ \vee(\text{state})\right),\text{ and}$$

- the post-measurement changes in logical entropy can be read off of the changes in the off-diagonal entries in the density matrices.

This all shows the logical structure of QM measurement in the simple context of "QM" over $\mathbb{Z}_2$.

6.4 Entanglement in "quantum mechanics" on sets

Another QM concept that also generates much mystery is entanglement. Hence it might be useful to consider entanglement in "quantum mechanics" on sets.

First we need to lift the set notion of the direct (or Cartesian) product $X\times Y$ of two sets X and Y. Using the basis principle, we apply the set concept to the two basis sets $\{v_1,...,v_m\}$ and $\{w_1,...,w_n\}$ of two vector spaces V and W (over the same base field) and then we see what it generates. The set direct product of the two basis sets is the set of all ordered pairs (v_i,w_j), which we will write as $v_i\otimes w_j$, and then we generate the vector space, denoted $V\otimes W$, over the same base field from those basis elements $v_i\otimes w_j$. That vector space is the *tensor product*, and it is not in general the direct product $V\times W$ of the vector spaces. The cardinality of $X\times Y$ is the product of the cardinalities of the two sets, and the dimension of the tensor product $V\otimes W$ is the product of the dimensions of the two spaces (while the dimension of the direct product $V\times W$ is the sum of the two dimensions).

A vector $z \in V \otimes W$ is said to be *separated* if there are vectors $v \in V$ and $w \in W$ such that $z = v \otimes w$; otherwise, z is said to be *entangled*. Since vectors delift to subsets, a subset $S \subseteq X \times Y$ is said to be *"separated"* or a *product* if there exists subsets $S_X \subseteq X$ and $S_Y \subseteq Y$ such that $S = S_X \times S_Y$; otherwise $S \subseteq X \times Y$ is said to be *"entangled."* In general, let S_X be the support or projection of S on X, i.e., $S_X = \{x : \exists y \in Y, (x, y) \in S\}$ and similarly for S_Y. Then S is "separated" iff $S = S_X \times S_Y$.

For any subset $S \subseteq X \times Y$, where X and Y are finite sets, a natural measure of its "entanglement" can be constructed by first viewing S as the support of the equiprobable or Laplacian joint probability distribution on S. If $|S| = N$, then define $\Pr(x, y) = \frac{1}{N}$ if $(x, y) \in S$ and $\Pr(x, y) = 0$ otherwise.

The marginal distributions[35] are defined in the usual way:

$$\Pr(x) = \sum_y \Pr(x, y)$$
$$\Pr(y) = \sum_x \Pr(x, y).$$

A joint probability distribution $\Pr(x, y)$ on $X \times Y$ is *independent* if for all $(x, y) \in X \times Y$,

$$\Pr(x, y) = \Pr(x) \Pr(y).$$

Independent distribution

Otherwise $\Pr(x, y)$ is said to be *correlated*.

For any correlated joint distribution $\Pr(x, y)$ on $X \times Y$ and any attribute $f : X \to \mathbb{R}$, we can extend the attribute to $f \times 1 : X \times Y \to \mathbb{R}$ by defining $f \times 1(x, y) = f(x)$ and then we have the average:

$$\bar{f} = \sum_x \Pr(x) f(x) = \sum_{x,y} \Pr(x, y) f \times 1(x, y).$$

But this should not be interpreted to mean that X "has" the distribution $\Pr(x)$ and that similarly Y "has" the distribution $\Pr(Y)$ sense then $X \times Y$ must "have" $\Pr(x) \Pr(y)$ as the joint distribution but the joint distribution $\Pr(x, y)$ is not independent by assumption. In the lifted case, the probabilities associated with a state in the tensor product of two systems might be represented by a pure state density matrix, and then the analogue of the marginal distributions would be the reduced density matrices which have the lifted version of the above averaging result. But then problems arise if one misinterprets the component systems as "being" in the states indicated by the reduced density matrices.[11, p. 61] In particular, those reduced density matrices are mixed and the product of mixed states is mixed rather than the assumed pure state on the tensor product. For an entangled composite system in a pure state represented by a density matrix, the component systems are not "in" the state represented by the reduced density matrices, just as when there is a non-independent probability distribution on a product of sets, then the component sets do not "have" the marginal distributions.

Proposition 1 *A subset $S \subseteq X \times Y$ is "entangled" iff the equiprobable distribution on S is correlated (non-independent).*

Proof: If S is "separated", i.e., $S = S_X \times S_Y$, then $\Pr(x) = |S_Y|/N$ for $x \in S_X$ and $\Pr(y) = |S_X|/N$ for $y \in S_Y$ where $|S_X||S_Y| = N$. Then for $(x, y) \in S$,

[35] The marginal distributions are the set versions of the reduced density matrices of QM.

$$\Pr(x,y) = \frac{1}{N} = \frac{N}{N^2} = \frac{|S_X||S_Y|}{N^2} = \Pr(x)\Pr(y)$$

and $\Pr(x,y) = 0 = \Pr(x)\Pr(y)$ for $(x,y) \notin S$ so the equiprobable distribution is independent. If S is "entangled," i.e., $S \neq S_X \times S_Y$, then $S \subsetneqq S_X \times S_Y$ so let $(x,y) \in S_X \times S_Y - S$. Then $\Pr(x), \Pr(y) > 0$ but $\Pr(x,y) = 0$ so it is not independent, i.e., is correlated. $\square$

Consider the set version of one qubit space where $U = \{a,b\}$. The product set $U \times U$ has 15 nonempty subsets. Each factor U and U has 3 nonempty subsets so $3 \times 3 = 9$ of the 15 subsets are "separated" subsets leaving 6 "entangled" subsets.

$S \subseteq U \times U$
$\{(a,a),(b,b)\}$
$\{(a,b),(b,a)\}$
$\{(a,a),(a,b),(b,a)\}$
$\{(a,a),(a,b),(b,b)\}$
$\{(a,b),(b,a),(b,b)\}$
$\{(a,a),(b,a),(b,b)\}$

The six entangled subsets

The first two are the "Bell states" which are the two graphs of bijections $U \longleftrightarrow U$ and have the maximum entanglement if entanglement is measured by the logical divergence $d(\Pr(x,y)||\Pr(x)\Pr(y))$[13]. All the 9 "separated" states have zero "entanglement" by the same measure.

For an "entangled" subset S, a sampling x of left-hand system will change the probability distribution for a sampling of the right-hand system y, $\Pr(y|x) \neq \Pr(y)$. In the case of maximal "entanglement" (e.g., the "Bell states"), when S is the graph of a bijection between U and U, the value of y is determined by the value of x (and vice-versa).

In this manner, we see that many of the basic ideas and relationships of quantum mechanical entanglement:

- "entangled states" and "separated" states,
- "reduced density matrices" (and the problems from misinterpreting them) ,
- maximally "entangled states," and
- "Bell states",

can be reproduced in "quantum mechanics" on sets.

The Bell Theorem for "quantum mechanics" over $\mathbb{Z}_2$ is developed in Appendix 4.

7 Waving good-by to waves

7.1 Wave-particle duality = indistinct-distinct particle 'duality'

States that are indistinct for an observable are represented as weighted vector sums or superpositions of the eigenstates that might be actualized by further distinctions. This indistinctness-represented-as-superpositions is usually interpreted as "wave-like aspects" of the particles in the indefinite state:

wave-like aspects of particle = indefiniteness in particle's state.

Hence the distinction-making measurements take away the indistinctness–which is usually interpreted as taking away the "wave-like aspects," i.e., "collapse of the wave packet." But there are no actual physical waves in quantum mechanics (e.g., the "wave amplitudes" are complex numbers); only particles with indistinct attributes for certain observables. Thus the "collapse of the wave packet" is better described as the "collapse of indefiniteness" to achieve definiteness. And the "wave-particle duality" is actually the contrast between particles in indefinite and definite states.

We have provided the back-story to objective indefiniteness by building the notion of distinctions from the ground up starting with partition logic and logical information theory. But the importance of distinctions and indistinguishability has been there all along in quantum mechanics.

Consider the standard double-slit experiment. When there is no distinction between the two slits, then the position attribute of the traversing particle is indefinite, neither top slit nor bottom slit (not "going through both slits"), which is usually interpreted as the "wave-like aspects" that show interference. But when a distinction is made between the slits, e.g., inserting a detector in one slit or closing one slit, then the distinction reduces the indefiniteness to definiteness so the indefiniteness disappears, i.e., the "wave-like aspects" such as interference disappear. For instance, Feynman makes this point about distinctions in terms of *distinguishing* the alternative final states.

> If you could, *in principle*, distinguish the alternative *final* states (even though you do not bother to do so), the total, final probability is obtained by calculating the *probability* for each state (not the amplitude) and then adding them together. If you *cannot* distinguish the final states *even in principle*, then the probability amplitudes must be summed before taking the absolute square to find the actual probability.[20, p. 3-9]

Moreover, when the properties of entities are carved out by distinctions (starting at the blob), then it is perfectly possible to have two entities that result from the same distinctions but with no other distinctions so they are *in principle* indistinguishable (unlike two twins who are "hard to tell apart"). In QM, this has enormous consequences as in the distinction between bosons and fermions, the Pauli exclusion principle, and the chemical properties of the elements. This sort of in-principle indistinguishability is a feature of the micro-reality envisaged by partition logic, but is not possible under the "properties all the way down" vision of the dual subset logic.

7.2 Wave math without waves = indistinctness-preserving evolution

The aim of the lifting program is to generate the mathematical framework of QM by lifting from partition logic and logical information theory. This program cannot generate any specific physical attribute-observables such as momentum or even space. But the general meta-physical idea (recall the "creation story" based on partition logic) is that there is some "substance" that is structured by the making of distinctions but is neither created nor destroyed. Hence we might surmise an amount-of-substance real-valued attribute that lifted to Hilbert space is represented by some Hermitian operator H and with some eigenvalues E and eigenvectors $|E\rangle$ as determined by the eigenvalue equation:

$$H\,|E\rangle = E\,|E\rangle.$$

"Substance" eigenvector equation

That is the mathematical form of the Schrödinger time-independent equation but the objective indefiniteness interpretation cannot tell us the form of the operator H (Hamiltonian) or the interpretation of the eigenvalues E (energy[36]).

What about the time-evolution of these constant-"substance" states? What about the (time-dependent) Schrödinger wave equation? Since measurements, or, more generally, interactions between a quantum system and the environment, may make distinctions (measurement and decoherence), we might ask the following question. What is the evolution of a quantum system that is *isolated* so that no distinctions are made in the sense that the degree of indistinctness between state vectors is not changed (in addition to the amount of "substance" being unchanged)? Two states ψ and φ in a Hilbert space are *fully distinct* if they are orthogonal, i.e., $\langle\psi|\varphi\rangle = 0$. Two states are *fully indistinct* if $\langle\psi|\varphi\rangle = 1$. In between, the *degree of indistinctness* can be measured by the *overlap* $\langle\psi|\varphi\rangle$, the inner product of the state vectors. Hence the evolution of an isolated quantum system where the degree of indistinctness does not change is described by a linear transformation that preserves inner products, i.e.,

transformations that preserve degree of indistinctness = unitary transformations.

The connection between unitary transformations and the solutions to the Schrödinger "wave" equation is given by Stone's Theorem [40]: there is a one-to-one correspondence between strongly continuous 1-parameter unitary groups $\{U_t\}_{t\in\mathbb{R}}$ and Hermitian operators H on the Hilbert space so that $U_t = e^{itH}$.

In simplest terms, a unitary transformation describes a rotation such as the rotation of the unit vector in the complex plane.

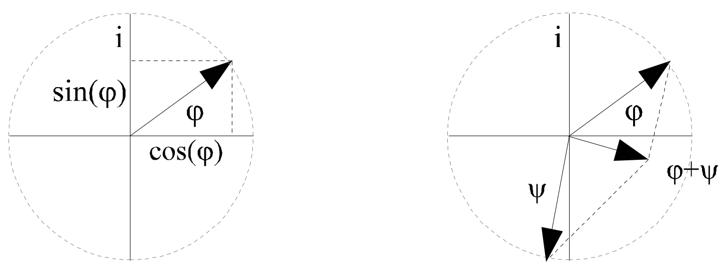


Figure 11: Rotating vector and addition of vectors

The rotating unit vector traces out the cosine and sine functions on the two axes, and the position of the arrow can be compactly described as a function of φ using Euler's formula:

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi).$$

[36] Heisenberg identifies the "substance" with energy.

> Energy is in fact the substance from which all elementary particles, all atoms and therefore all things are made, and energy is that which moves. Energy is a substance, since its total amount does not change, and the elementary particles can actually be made from this substance as is seen in many experiments on the creation of elementary particles.[24, p. 63]

Such complex exponentials and their superpositions are the "wave functions" of QM. The "wave functions" describe the evolution of particles in indefinite states in isolated systems where there are no interactions to change the degree of indistinctness between states, i.e., the context where Schrödinger's equation holds. Classically it has been assumed that the mathematics of waves must describe physical waves of some sort, and thus the puzzlement about the "waves" of QM having complex amplitudes and no corresponding physical waves. But we have supplied *another* interpretation; wave mathematics is the mathematics of indefiniteness, e.g., superposition represents indefiniteness and unitary evolution represents the indistinctness-preserving evolution of an isolated system. Feynman's addition of turning arrows [19] is an example of building an imagery different from the usual wave imagery.

Thus the objective indefiniteness approach to interpreting QM provides an explanation for the appearance of the wave mathematics (which implies interference as well as the quantized solutions to the "wave" equation that gave QM its name) when, in fact, there are no actual physical waves involved.

8 Logical entropy measures measurement

8.1 Logical entropy as the total distinction probability

The notion of logical entropy of a probability distribution $p = (p_1, ..., p_n)$, $h(p) = 1 - \sum_i p_i^2$, generalizes to the *quantum logical entropy* of a density matrix ρ [16],

$$h(\rho) = 1 - \text{tr}\left[\rho^2\right].$$

Given a state vector $|\psi\rangle = \sum_i \alpha_i |i\rangle$ expressed in the orthonormal basis $\{|i\rangle\}_{i=1,...,n}$, the density matrix

$$\rho(\psi) = |\psi\rangle\langle\psi| = [\rho_{ij}] = \left[\alpha_i \alpha_j^*\right]$$

(where α_j^* is the complex conjugate of α_j) is a *pure state* density matrix. For a pure state density matrix:

$$h(\rho) = 1 - \text{tr}\left[\rho^2\right] = 1 - \sum_i \sum_j \alpha_i \alpha_j^* \alpha_j \alpha_i^* = 1 - \sum_i \alpha_i \alpha_i^* \sum_j \alpha_j \alpha_j^* = 1 - 1 = 0.$$

Otherwise, a density matrix ρ is said to represent a *mixed state*, and its logical entropy is positive.

In the set case, the logical entropy $h(\pi)$ of a partition π was interpreted as the probability that two independent draws from U (equiprobable elements) would give a distinction of π. For a probability distribution $p = (p_1, ..., p_n)$, the logical entropy $h(p) = 1 - \sum_i p_i^2$ is the probability that two independent samples from the distribution will give distinct outcomes $i \neq j$. The probability of the distinct outcomes (i, j) for $i \neq j$ is $p_i p_j$. Since $1 = (p_1 + ... + p_n)(p_1 + ... + p_n) = \sum_{i,j} p_i p_j$, we have:

$$h(p) = 1 - \sum_i p_i^2 = \sum_{i,j} p_i p_j - \sum_i p_i^2 = \sum_{i \neq j} p_i p_j$$

which is the sum of all the distinction (i.e., distinct indices) probabilities.

This interpretation generalizes to the quantum logical entropy $h(\rho)$. The diagonal terms $\{p_i\}$ in a density matrix:

$$\rho = \begin{bmatrix} p_1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & p_2 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & p_n \end{bmatrix}$$

are the probabilities of getting the i^{th} eigenvector $|i\rangle$ in a projective measurement of a system in the state ρ (using $\{|i\rangle\}$ as the measurement basis). The off-diagonal terms ρ_{ij} give the amplitude that the eigenstates $|i\rangle$ and $|j\rangle$ cohere, i.e., are indistinct, in the state ρ so the absolute square $|\rho_{ij}|^2$ is the *indistinction probability*. Since $p_i p_j$ is the probability of getting $|i\rangle$ and $|j\rangle$ in two independent measurements, the difference $p_i p_j - |\rho_{ij}|^2$, is the *distinction probability*. But $1 = \sum_{i,j} p_i p_j$ so we see that the interpretation of the logical entropy as the total distinction probability carries over to the quantum case:

$$h(\rho) = 1 - \operatorname{tr}\left[\rho^2\right] = 1 - \sum_{ij} |\rho_{ij}|^2 = \sum_{ij} \left[p_i p_j - |\rho_{ij}|^2\right] = \sum_{i \neq j} \left[p_i p_j - |\rho_{ij}|^2\right]$$

Quantum logical entropy = sum of distinction probabilities

where the last step follows since $p_i p_i - |\rho_{ii}|^2 = 0$.

8.2 Measuring measurement

Since $h(\rho) = 0$ for a pure state ρ, that means that all the eigenstates $|i\rangle$ and $|j\rangle$ cohere together, i.e., are indistinct, with various amplitudes in a pure state, like the indiscrete partition in the set case. For set partitions, the transition, $\mathbf{0} \to \mathbf{1} \vee \mathbf{0} = \mathbf{1}$, from the indiscrete to the discrete partition turns all the indistinctions (i,j) (where $i \neq j$) into distinctions, and the logical entropy increases from 0 to $1 - \sum_i p_i^2 = 1 - \frac{1}{n}$ where $p_i = \frac{1}{n}$ for $|U| = n$.

In quantum mechanics, an observable with non-degenerate eigenvalues has a vector space partition $\mathbf{1}$ with one-dimensional eigenspaces. That vector space partition (analogous to the set partition with singleton blocks $\left\{f^{-1}(r)\right\} = \mathbf{1}$) might also be represented by a set of one-dimensional projection operators $P_{\lambda(i)}$ (analogous to the set-projection operators $\left\{f^{-1}(r) \cap ()\right\}$). As a matrix in the measurement basis, $P_{\lambda(i)}$ would have a 1 in the i^{th} place on the diagonal and 0's elsewhere . The action of the measurement is represented by the action of the partition of projection operators on the given pure state ψ. In terms of density matrices, the non-degenerate measurement turns the pure state density matrix $\rho(\psi) = |\psi\rangle\langle\psi|$ into the mixed state diagonal matrix $\hat{\rho}(\psi) = \rho(\mathbf{1}) = \sum_{i=1}^{n} P_{\lambda(i)} \rho P_{\lambda(i)}$ with the same diagonal entries $p_1, ..., p_n$:

$$\rho(\psi) = \begin{bmatrix} p_1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & p_2 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & p_n \end{bmatrix} \Longrightarrow \hat{\rho}(\psi) = \sum_{i=1}^{n} P_{\lambda(i)} \rho P_{\lambda(i)} = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_n \end{bmatrix}.$$

A non-degenerate measurement

Hence the quantum logical entropy similarly goes from $h(\rho) = 0$ to $h(\hat{\rho}) = 1 - \sum_i p_i^2$. This is usually described by saying that all the off-diagonal coherence terms are decohered in a nondegenerate measurement–which means that all the indistinctions $(|i\rangle, |j\rangle)$ where $|i\rangle \neq |j\rangle$ of the pure state are distinguished by the measurement. And the sum of all those new distinction probabilities for the decohered off-diagonal terms is precisely the quantum logical entropy since $h(\hat{\rho}) = \sum_{i \neq j} \left[p_i p_j - |\hat{\rho}_{ij}|^2 \right] = \sum_{i \neq j} p_i p_j = 1 - \sum_i p_i^2$. For any measurement (degenerate or not), the increase in logical entropy

$$h(\hat{\rho}) - h(\rho) = \sum_{new} |\rho_{ij}|^2 = \text{sum of } \textit{new} \text{ distinction probabilities}$$

where the sum is over the zeroed or decohered coherence terms $|\rho_{ij}|^2$ that gave indistinction probabilities in the pre-measurement state ρ. Thus we see how quantum logical entropy interprets the off-diagonal entries in the state density matrices and how the change in the quantum logical entropy measures precisely the decoherence, i.e., the distinctions, made by a measurement.[37]

9 Lifting to the axioms of quantum mechanics

We have now reached the point where the program of lifting partition logic and logical information theory to the quantum concepts of Hilbert spaces essentially yields the abstract axioms of quantum mechanics.

Using axioms based on [34], the first axiom gives the vector space endpoint of the lifting program.

Axiom 1: *An isolated system is represented by a complex inner product vector space (i.e., a Hilbert space) where the complete description of a state of the system is given by a state vector, a unit vector in the system's space.*

Two fully distinct states would be orthogonal (thinking of them as eigenstates of an observable), and a state indefinite between them would be represented as a weighted vector sum or superposition of the two states. Taking a superposition state as a "complete description" is essentially the same as saying that the indefiniteness is objective.

We previously saw that the evolution of a closed system that preserves the degree of indistinction between states would be a unitary transformation.

Axiom 2: *The evolution of a closed quantum system is described by a unitary transformation.*

In the last section, we saw how a projective measurement would zero some or all of the off-diagonal coherence terms in a pure state ρ to give a mixed state $\hat{\rho}$ (and how the sum of the absolute squares of the zeroed coherence terms gave the change in quantum logical entropy).

Axiom 3: *A projective measurement for an observable (Hermitian operator) $F = \sum_{i=1}^{n} \lambda(i) P_{\lambda(i)}$ (spectral decomposition using projection operators $P_{\lambda(i)}$) on a pure state ρ has an outcome $\lambda(i)$ with probability $p_i = \rho_{ii}$ giving the mixed state $\hat{\rho} = \sum_{i=1}^{n} P_{\lambda(i)} \rho P_{\lambda(i)}$.*

And finally we saw how the basis principle lifted the notion of combining sets with the direct product of sets $X \times Y$ to the notion of representing combined quantum systems with the vector space generated by the direct product of two basis sets of the state spaces, i.e., the tensor product.

Axiom 4: *The state space of a composite system is the tensor product of the state spaces of the component systems.*

[37] In contrast, the standard notion of entropy currently used in quantum information theory, the von Neumann entropy, is only qualitatively related to measurement, i.e., projective measurement increases von Neumann entropy [34, p. 515].

10 Conclusion

The objective indefiniteness interpretation of quantum mechanics is based on using partition logic, logical information theory, and the lifting program to fill out the back story to the old notion of "objective indefiniteness" ([37], [38]). In Appendix 1, the lifting program is further applied to lift set representations of groups to vector space representations, and thus to explain the fundamental importance of group representation theory in quantum mechanics (not to mention particle physics). In Appendices 2, 3, 4, and 5, the Heisenberg indefiniteness principle, the two-slit experiment, Bell's Theorem, and the measurement problem are treated in "quantum mechanics" over $\mathbb{Z}_2$–which lays out the bare logical structure in all the cases.

At the level of sets, if we start with a universe set U as representing our common-sense macroscopic world, then there are *only two* logics, the logics of subsets and quotient sets (i.e., partitions), to envisage the "creation story" for U. Increase the size of subsets or increase the refinement of quotient sets until reaching the universe U. That is, starting with the empty subset of U, take larger and larger subsets of fully definite elements until finally reaching all the fully definite elements of U. Or starting with the indiscrete partition on U, take more and more refined partitions, each block interpreted as an indefinite element, until finally reaching all the fully definite elements of U. Those are the two dual ways to "create" U.

Classical physics was compatible with the subset creation story in the sense that the elements were always fully propertied ("properties all the way down"). But almost from the beginning, quantum mechanics was seen not to be compatible with that world view of always fully definite entities; QM seems to envisage entities at the micro-level that are objectively indefinite. Within the framework of the two logics given by subset-partition duality, the "obvious" thing to do is to elaborate on the dual creation story to try to build *the* other interpretation of QM.

With the development of the logic of partitions (dual to the logic of subsets) and the quantitative measure of partitions in logical information theory, the foundation was in place to lift those set concepts to the richer mathematical environment of complex vector spaces with inner products. In that manner, *the* other interpretation of QM was constructed. All that–partition logic, logical information theory, and the lifting program to the linearized partition mathematics of vector spaces–is just mathematics, a mathematical description of a world of objectively indefinite entities. Then comes the physics. We note that the result reproduces the basic ideas and mathematical machinery of the physical theory of quantum mechanics, e.g., as expressed in four axioms given above. And starting with QM, the reverse delifting program creates the non-metrical *logic of QM* over $\mathbb{Z}_2$. That completes an outline of the objective indefiniteness interpretation of quantum mechanics.

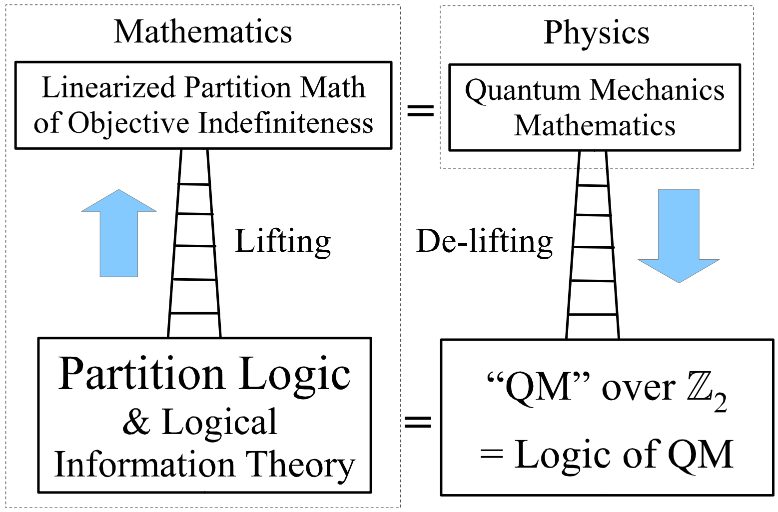


Figure 11a: Outline of objective indefiniteness interpretation of quantum mechanics

11 Appendix 1: Lifting in group representation theory

11.1 Group representations define partitions

Given a *set* G indexing mappings $\{R_g : U \to U\}_{g\in G}$ on a set U, what are the conditions on the set of mappings so that it is a set representation of a group? Define the binary relation on $U \times U$:

$$u \sim u' \text{ if } \exists g \in G \text{ such that } R_g(u) = u'.$$

Then the conditions that make R into a group representation are the conditions that imply $u \sim u'$ is an equivalence relation:

1. existence of the identity $1_U \in U$ implies reflexivity of $\sim$;
2. existence of inverses implies symmetry of $\sim$; and
3. closure under products, i.e., for $g, g' \in G$, $\exists g'' \in G$ such that $R_{g''} = R_{g'}R_g$, implies transitivity of $\sim$.

Hence a set representation of a group might be seen as a "dynamic" way to define an equivalence relation and thus a partition on the set.[5] A symmetry group defines indistinctions. For instance, if linear translations form a symmetry group for a quantum system, then the system behavior before a linear translation is indistinct from the behavior of the translated system. Given this intimate connection between groups and partitions, it is then no surprise that group representation theory has a basic role to play in quantum mechanics and in the partition-based objective indefiniteness interpretation of QM.

11.2 Where do the fully distinct eigen-alternatives come from?

In classical mechanics, the role of symmetry groups is to establish invariances, e.g., Noether's Theorem. But in quantum mechanics, the spaces satisfy the superposition principle (i.e., objectively indefinite superpositions) and that allows symmetry groups and group representation theory to play a much more fundamental role than simply the role of accounting for invariants in classical mechanics. What is that more fundamental role that goes beyond symmetry-induced invariance? The more fundamental role is to determine–within the constraints of the symmetries–what are all the maximally distinct eigen-alternatives.

In a quantum state space, we are *given* the observable with its distinct eigenstates so the indefinite states are linear combinations of those eigenstates. But how is the observable with the range of distinct eigen-alternatives determined?

In the set case, we are *given* the universe U of distinct eigen-alternatives $u \in U$, and then the indistinct entities are the subsets such as the blocks $B \in \pi$ in a partition of U. A "measurement" is some distinction-making operation that reduces an indistinct state B down to a more distinct state $B' \subseteq B$ or, in the nondegenerate case, to a fully distinct singleton $\{u\}$ for some $u \in B$. But where do the fully distinct elements come from?

The *basic idea* is that a symmetry group defines indistinctions, so what are all the ways that there can be distinct eigen-elements that are consistent with those indistinctions? In a representation of a group by permutations on a set U, the answer is:

distinct eigen-elements consistent with symmetry group $\approx$ orbits of group representation.

Two elements of U inside the same orbit cannot be considered distinct in a way consistent with the indistinction-making action of the group since they are, by definition, mapped from one to the other by an "indistinction-making" group operation. Hence the maximally distinct subsets–consistent with the indistinction-making symmetries–are the minimal invariant subsets, the orbits in the set case. That is how the partition ideas mesh with group representation theory. First we consider the set version, and then we lift to the vector space version of group representation theory.

Let U be a set and $S(U)$ the group of all permutations of U. Then a *set representation* of a group G is an assignment $R : G \rightarrow S(U)$ where for $g \in G$, $g \longmapsto R_g \in S(U)$ such that R_1 is the identity on U and for any $g, g' \in G$, $R_{g'}R_g = R_{g'g}$. Equivalently, a *group action* is a binary operation $G \times U \rightarrow U$ such that $1u = u$ and $g'(gu) = (g'g)u$ for all $u \in U$.

Defining $u \sim u'$ if $\exists g \in G$ such that $R_g(u) = u'$ [or $gu = u'$ using the group action notation], we have an equivalence relation on U where the blocks are called the *orbits*.

How are the ultimate distinct eigen-alternatives, the distinct "eigen-forms" of "substance," defined in the set case? Instead of just assuming U as the set of eigen-alternatives, we start with U as the carrier for a set representation of the group G as a group of symmetries. What are the smallest subsets (forming the blocks B in a set partition) that respect the symmetries, i.e., that are *invariant* in the sense that $R_g(B) \subseteq B$ for all $g \in G$? Those minimal invariant subsets are the orbits, and all invariant subsets are unions of orbits. Thus the orbits, thought of as points in the quotient set U/G (set of orbits), are the eigen-alternatives, the "eigen-forms" of "substance," defined by the symmetry group G in the set case.

Example 1: Let $U = \{0,1,2,3,4,5\}$ and let $G = S_2 = \{1,\sigma\}$ (symmetric group on two elements) where $R_1 = 1_U$ and $R_\sigma(u) = u + 3 \bmod 6$.

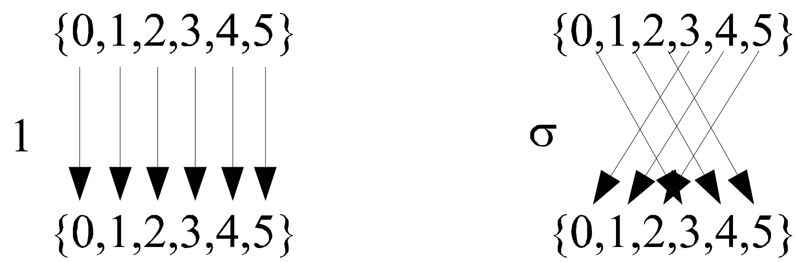


Figure 12: Action of S_2 on $U = \{0, 1, 2, 3, 4, 5\}$

There are 3 orbits: $\{0, 3\}$, $\{1, 4\}$, and $\{2, 5\}$, and they partition U. Those three orbits are the "points" in the quotient set U/G, i.e., they are the distinct eigen-alternatives defined by the symmetry group's S_2 action on U.

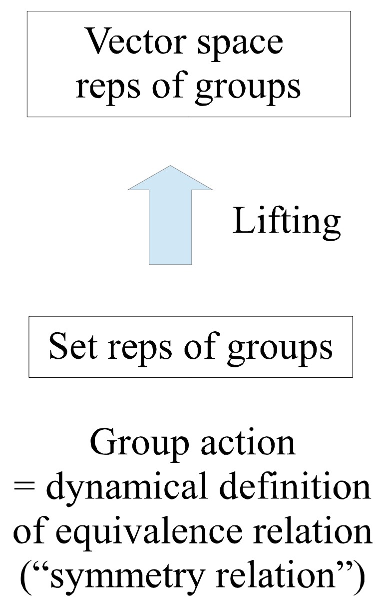


Figure 12a: Lifting from set reps to vector space reps of groups

A *vector space representation* of a group G on a vector space V is a mapping $g \longmapsto R_g : V \to V$ from G to invertible linear transformations on V such that $R_{g'} R_g = R_{g'g}$.

The lifts to the vector space representations of groups are;

- minimal invariant subsets = orbits $\xrightarrow{Lifts}$ minimal invariant subspaces = *irreducible subspaces*,
- representation restricted to orbits $\xrightarrow{Lifts}$ representation restricted to irreducible subspaces which gives the *irreducible representations* (the eigen-forms of substance in the vector space case[38]), and
- set partition of orbits $\xrightarrow{Lifts}$ vector space partition of irreducible subspaces.

[38] In Heisenberg's philosophical terms, the irreducible representations of certain symmetry groups of particle physics determine the fundamental eigen-forms that the substance (energy) can take.

The "irreducible representations" in the set case are just the restrictions of the representation to the orbits, e.g., $R \upharpoonright \{0,3\} : S_2 \to S(\{0,3\})$, as their carriers. A set representation is said to be *transitive*, if for any $u, u' \in U$, $\exists g \in G$ such that $R_g(u) = u'$. A transitive set representation has only one orbit, all of U. Any set "irreducible representation" is transitive.

We are accustomed to thinking of some distinction-making operation as reducing a whole partition to a more refined partition, and thus breaking up a block B into distinguishable non-overlapping subsets $B', B'', ... \subseteq B$. Now we are working at the more basic level of determining the distinct eigen-alternatives, i.e., the orbits of a set representation of a symmetry group. Here we might also consider how distinctions are made to move to a more refined partition of orbits. Since the group operations identify elements, $u \sim u'$ if $\exists g \in G$ such that $R_g(u) = u'$, we would *further distinguish* elements by moving to a subgroup, i.e., fewer group elements making indistinctions so more distinctions in maximally distinct eigen-alternatives. The symmetry operations in the larger group are "broken," so the remaining group of symmetries is a subgroup. That is how *symmetry-breaking* is accounted for in this interpretation.

Example 1 revisited: the group S_2 has only one subgroup, the trivial subgroup of the identity operation, and its orbits are clearly the singletons $\{u\}$ for $u \in U$. That is the simplest example of symmetry-breaking that gives a more distinct set of eigen-alternatives.

In any set representation, the maximum distinctions are made by the smallest symmetry subgroup which is always the identity subgroup, so that is always the waste case that takes us back to the singleton orbits in U, i.e., the distinct elements of U.

Thus we see that symmetry-breaking is analogous to distinction-making measurement but at this more fundamental level where the distinct eigen-forms are determined in the first place by symmetries.

11.3 Attributes and observables

An (real-valued) *attribute* on a set U is a function $f : U \to \mathbb{R}$. An attribute induces a set partition $\{f^{-1}(r)\}$ on U. An attribute $f : U \to \mathbb{R}$ *commutes* with a set representation $R : G \to S(U)$ if for any R_g, the following diagram commutes in the sense that $fR_g = f$:

$$\begin{array}{ccc} U & \xrightarrow{R_g} & U \\ & \searrow^{f} & \downarrow^{f} \\ & & \mathbb{R} \end{array}$$

Commuting attribute.

The lifts to vector space representations are immediate:

- a real-valued attribute on a set $\xrightarrow{Lifts}$ an observable represented by a Hermitian operator on a complex vector space; and
- the commutativity condition on a set-attribute $\xrightarrow{Lifts}$ an observable operator H (like the Hamiltonian) commuting with a symmetry group in the sense that $HR_g = R_gH$ for all $g \in G$.

The elementary particles are therefore the fundamental forms that the substance energy must take in order to become matter, and these basic forms must in some way be determined by a fundamental law expressible in mathematical terms. ... The real conceptual core of the fundamental law must, however, be formed by the mathematical properties of the symmetry it represents.[25, pp. 16-17]

The commutativity-condition in the set case means that whenever $R_g(u) = u'$ then $f(u) = f(u')$, i.e., that f is an *invariant* of the group. Recall that each orbit of a set representation is transitive so for any u, u' in the same orbit, $\exists R_g$ such that $R_g(u) = u'$ so $f(u) = f(u')$ for any two u, u' in the same orbit. In other words:

"**Schur's Lemma"** (set version): a commuting attribute restricted to an orbit is constant.

The lift to vector space representations is one version of the usual

Schur's Lemma (vector space version): An operator H commuting with G restricted to irreducible subspace is a constant operator.

This also means that the inverse-image partition $\{f^{-1}(r)\}$ of a commuting attribute is refined by the orbit partition. If an orbit $B \subseteq f^{-1}(r)$, then the "eigenvalue" r of the attribute f is associated with that orbit. Every commuting attribute $f : U \to \mathbb{R}$ can be uniquely expressed as a "spectral decomposition":

$$f = \sum_{o \in Orbits} r_o \chi_o,$$

where r_o is the constant value on the orbit $o \subseteq U$ and $\chi_o : U \to \mathbb{R}$ is the characteristic function of the orbit o.

There may be other orbits with the same "eigenvalue." Then we would need another commuting attribute $g : U \to \mathbb{R}$ so that for each orbit B, there is an "eigenvalue" s of the attribute g such that $B \subseteq g^{-1}(s)$. Then the eigen-alternative B may be characterized by the ordered pair $|r, s\rangle$ if $B = f^{-1}(r) \cap g^{-1}(s)$. If not, we continue until we have a *Complete Set of Commuting Attributes* (CSCA) whose ordered n-tuples of "eigenvalues" would characterize the eigen-alternatives, the orbits of the set representation $R : G \to S(U)$.

Obviously, we are just spelling out the set version whose lift is the use of a Complete Set of Commuting Operators (CSCO) to characterize the eigenstates by kets of ordered n-tuples $|\lambda, \mu, ...\rangle$ of eigenvalues of the commuting operators. *But* these "eigenstates" are not the singletons $\{u\}$ but are the maximally distinct invariant subsets or orbits of the set representation of the symmetry group G. The basic theme is that the indefinite elements are distinguished to form more definite entities by the distinctions made by the joins of the inverse-image partitions of compatible attributes.

Example 1 again: Consider the attribute $f : U = \{0, 1, 2, 3, 4, 5\} \to \mathbb{R}$ where $f(n) = n \bmod 3$. This attribute commutes with the previous set representation of S_2, namely $R_1 = 1_U$ and $R_\sigma(u) = u + 3 \bmod 6$, and accordingly by "Schur's Lemma" (set version), the attribute is constant on each orbit $\{0, 3\}, \{1, 4\}$, and $\{2, 5\}$. In this case, the blocks of the inverse-image partition $\{f^{-1}(0), f^{-1}(1), f^{-1}(2)\}$ equal the blocks of the orbit partition, so this attribute is the set version of a "nondegenerate measurement" in that its "eigenvalues" suffice to characterize the eigen-alternatives, i.e., the orbits. By itself, it forms a complete set of attributes.

Example 2: Let $U = \{0, 1, ..., 11\}$ where $S_2 = \{1, \sigma\}$ is represented by the operations $R_1 = 1_U$ and $R_\sigma(n) = n + 6 \bmod (12)$. Then the orbits are $\{0, 6\}, \{1, 7\}, \{2, 8\}, \{3, 9\}, \{4, 10\}$, and $\{5, 11\}$. Consider the attribute $f : U \to \mathbb{R}$ where $f(n) = n \bmod (2)$. This attribute commutes with the symmetry group and is thus constant on the orbits. But the blocks in the inverse-image partition are now larger than the orbits, i.e., $f^{-1}(0) = \{0, 2, 4, 6, 8, 10\}$ and $f^{-1}(1) = \{1, 3, 5, 7, 9, 11\}$ so the orbit partition strictly refines $\{f^{-1}(r)\}$. Thus this attribute corresponds to a degenerate measurement in that the two "eigenvalues" do not suffice to characterize the orbits.

Consider the attribute $g : U \to \mathbb{R}$ where $g(n) = n \bmod (3)$. This attribute commutes with the symmetry group and is thus constant on the orbits. The blocks in the inverse-image partition are:

$g^{-1}(0) = \{0,3,6,9\}$, $g^{-1}(1) = \{1,4,7,10\}$, and $g^{-1}(2) = \{2,5,8,11\}$. The blocks in the join of the two partitions $\{f^{-1}(r)\}$ and $\{g^{-1}(s)\}$ are the non-empty intersections of the blocks:

$f^{-1}(r)$	$g^{-1}(s)$	$f^{-1}(r) \cap g^{-1}(s)$	$\|r,s\rangle$
$\{0,2,4,6,8,10\}$	$\{0,3,6,9\}$	$\{0,6\}$	$\|0,0\rangle$
$\{0,2,4,6,8,10\}$	$\{1,4,7,10\}$	$\{4,10\}$	$\|0,1\rangle$
$\{0,2,4,6,8,10\}$	$\{2,5,8,11\}$	$\{2,8\}$	$\|0,2\rangle$
$\{1,3,5,7,9,11\}$	$\{0,3,6,9\}$	$\{3,9\}$	$\|1,0\rangle$
$\{1,3,5,7,9,11\}$	$\{1,4,7,10\}$	$\{1,7\}$	$\|1,1\rangle$
$\{1,3,5,7,9,11\}$	$\{2,5,8,11\}$	$\{5,11\}$	$\|1,2\rangle$

f and g as a complete set of commuting attributes

Thus f and g form a Complete Set of Commuting Attributes to characterize the eigen-alternatives, the orbits, by the "kets" of ordered pairs of their "eigenvalues."

Example 3: Let $U = \mathbb{R}^2$ as a set and let G be the special orthogonal matrix group $SO(2,\mathbb{R})$ of matrices of the form;

$$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \text{ for } 0 \le \varphi < 2\pi.$$

This group is trivially represented by the rotations in $U = \mathbb{R}^2$:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The orbits are the circular orbits around the origin. The attribute "radius" $f : \mathbb{R}^2 \to \mathbb{R}$ where $f(x,y) = \sqrt{x^2+y^2}$ commutes with the representation since:

$f(x',y') = \sqrt{(x')^2 + (y')^2}$
$= \sqrt{(x\cos\varphi - y\sin\varphi)^2 + (x\sin\varphi + y\cos\varphi)^2}$
$= \sqrt{x^2\left(\cos^2\varphi + \sin^2\varphi\right) + y^2\left(\cos^2\varphi + \sin^2\varphi\right)}$
$= f(x,y)$.

That means that "radius" is an invariant of the rotation symmetry group. The blocks in the set partition $\{f^{-1}(r) : 0 \le r\}$ of $\mathbb{R}^2$ coincide with the orbits so the "eigenvalues" of the radius attribute suffice to characterize the orbits.

Example 4: The Cayley set representation of any group G is given by permutations on $U = G$ itself defined by $R_g(g') = gg'$, which is also called the *left regular representation*. Given any $g, g' \in G$, $R_{g'g^{-1}}(g) = g'$ so the Cayley representation is always transitive, i.e., has only one orbit consisting of all of $U = G$. Since any commuting attribute $f : U = G \to \mathbb{R}$ is constant on each orbit, it can only be a constant function such as χ_G.

Thus the Cayley set representation is rather simple, but we could break some symmetry by considering a proper subgroup $H \subseteq G$. Then using only the R_h for $h \in H$, we have a representation $H \to S(G)$. The orbit-defining equivalence relation is $g \sim g'$ if $\exists h \in H$ such that $hg = g'$, i.e., the orbits are the *right cosets* Hg.

The lifting program from set representations to vector space representations is summarized in the following table and will be illustrated in the next section.

Lifting Program	Set group representations	Vector space group reps
Representation	Group G represented by permutations R_g:U→U	Group G represented by invertible linear ops. R_g:V→V
Min. invariants	Orbits	Irreducible subspaces
Partition	Set partition of orbits	Vector space partition of irreducible subspaces
Irreducible reps	Reps restricted to orbits	Reps restricted to irred. spaces
Commuting with representation	Attribute f:U→ℝ commuting with R_g, i.e., $fR_g = f$.	Operator H commuting with R_g, i.e., $HR_g = R_gH$
Invariants	Inverse-images $f^{-1}(r)$ for commuting f are invariant.	Eigenspaces of commuting H are invariant.
Schur's Lemma	Commuting f restricted to orbit is constant function.	Commuting H restricted to irred. subspace is constant op.

Figure 13: Lifting program for group representations

11.4 Irreps of vector space representations

Our conceptual purpose here is to describe how group representation theory, the set version or the lift to vector spaces, answers the question of determining the form of the distinct eigen-alternatives. Given a group of symmetries acting on a set or on a vector space, what are the *most distinct* subsets or subspaces that satisfy the symmetries? Those minimal invariant subobjects are the orbits in the set case and the irreducible subspaces in the vector space case. In each case, those subobjects give the appropriate type of partition (i.e., a set partition or a vector space partition). And from partition logic, we know that the way to carve out more refined alternatives is the join of partitions. We have already illustrated this in the set case where the inverse-image partitions of a complete set of compatible attributes were joined so that they characterized the most distinct eigen-forms, the orbits of the representation. We will illustrate how the same partition methods of CSCOs apply in the lifted case of vector space representations.

The distinct eigen-alternatives in the set case (the orbits) cannot show much variation since there is only the question of an element of the universe set U being in or out of a subset. That is, viewing a subset $S \subseteq U$ as a vector in $\mathbb{Z}_2^{|U|}$, the coefficients of each basis vector are only 0 or 1. But when we lift to group representations over a vector space with $\mathbb{C}$ as the base field, then there is much more variation in distinct eigen-alternatives. And it is that much wider range of maximally-distinct eigen-forms, the irreducible representations that are the representations restricted to the irreducible subspaces, that are of such importance in quantum mechanics.

The partition-join method of determining the eigen-forms for vector space representations will be illustrated with several simple examples with finite dimensional vector spaces V. Recall that a *vector space representation* of a group G is given by an assignment of an invertible linear operator $R_g : V \to V$ to each $g \in G$ such that $R_1 = 1_V$ and $R_{g'}R_g = R_{g'g}$.

Example 5: The multiplicative group $S_2 \times S_2$ written additively is the Klein four-group $G =$

$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (1,0), (0,1), (1,1)\}$. The complex vector space $\{\mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{C}\}$ of all complex-valued functions on the four-element set $\mathbb{Z}_2 \times \mathbb{Z}_2$ is the *Cayley group space* of that group. A basis for the four-dimensional space $\mathbb{C}^4$ is the set of functions $|g'\rangle$ which take value 1 on g' and 0 on the other $g \in G$. Then the action of the group on this space is defined by $R_g(|g'\rangle) = |g+g'\rangle$ (or $|gg'\rangle$ if the group operation was written multiplicatively). Thus the group action just permutes the basis vectors in the Cayley group space and would be represented by permutation matrices. The non-identity operators have the matrices;

$$R_{(1,0)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} (0,0) \\ (1,0) \\ (0,1) \\ (1,1) \end{matrix} ; R_{(0,1)} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} ; R_{(1,1)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Since the group is Abelian, each of these operators can be viewed as an observable H that commutes with the R_g for $g \in G$, so its eigenspaces will be invariant under the group operations.

For $R_{(1,0)}$, the invariant eigenspaces with their eigenvalues and generating eigenvectors are:

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\} \leftrightarrow \lambda = -1, \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\} \leftrightarrow \lambda = 1.$$

For $R_{(0,1)}$, we have:

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\} \leftrightarrow \lambda = -1, \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\} \leftrightarrow \lambda = 1.$$

Since these two operators commute, their eigenspace partitions are compatible so we can take their join. The blocks of the join are a vector space partition and are automatically invariant. Since the blocks of the join are one-dimensional, those four subspaces are also irreducible and thus those two operators form a complete set of commuting operators (CSCO). The commuting operators always have a set of simultaneous eigenvectors, and we have arranged the generating eigenvectors of the eigenspaces so that they are all simultaneous eigenvectors which can, as usual, be characterized by kets using the respective eigenvalues;

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = |1,1\rangle; \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = |-1,1\rangle; \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = |1,-1\rangle; \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = |-1,-1\rangle.$$

The group representation restricted to these four irreducible subspaces give the four irreducible representations or irreps of the group. Since any vector can be uniquely decomposed into the sum of vectors in the irreducible subspaces, the representation on the whole space can be expressed, in the obvious sense, as the direct sum of the irreps.

In the set case, moving to smaller invariant subsets by making distinctions gives more distinct elements, so in the vector space case, moving to smaller invariant subspaces give more distinct alternatives. The minimal invariant subspaces, i.e., the irreducible subspaces, thus give the maximally-distinct invariant subspaces, and the representation restricted to those subspaces gives the maximally-distinct symmetry-respecting alternatives, i.e., the irreps.

This might be illustrated by giving a geometric version of the representation. Consider a rectangle $\boxed{\begin{smallmatrix}a&b\\d&c\end{smallmatrix}}$ under the operations of flipping on the horizontal axis $\sigma_h : \boxed{\begin{smallmatrix}a&b\\d&c\end{smallmatrix}} \longmapsto \boxed{\begin{smallmatrix}d&c\\a&b\end{smallmatrix}}$ and flipping on the vertical axis: $\sigma_v : \boxed{\begin{smallmatrix}a&b\\d&c\end{smallmatrix}} \longmapsto \boxed{\begin{smallmatrix}b&a\\c&d\end{smallmatrix}}$ as well as their composition $\sigma_{hv} : \boxed{\begin{smallmatrix}a&b\\d&c\end{smallmatrix}} \longmapsto \boxed{\begin{smallmatrix}c&d\\b&a\end{smallmatrix}}$ and the identity. This gives the same $S_2 \times S_2$ group with the multiplication table:

2nd \ 1st	1	σ_h	σ_v	σ_{hv}
1	$\boxed{\begin{smallmatrix}a&b\\d&c\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}d&c\\a&b\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}b&a\\c&d\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}c&d\\b&a\end{smallmatrix}}$
σ_h	$\boxed{\begin{smallmatrix}d&c\\a&b\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}a&b\\d&c\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}c&d\\b&a\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}b&a\\c&d\end{smallmatrix}}$
σ_v	$\boxed{\begin{smallmatrix}b&a\\c&d\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}c&d\\b&a\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}a&b\\d&c\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}d&c\\a&b\end{smallmatrix}}$
σ_{hv}	$\boxed{\begin{smallmatrix}c&d\\b&a\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}b&a\\c&d\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}d&c\\a&b\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}a&b\\d&c\end{smallmatrix}}$

Multiplication table for $S_2 \times S_2$ as symmetry group
resulting from flipping on the horizontal and vertical axes.

Then the basis vectors for the four irreducible subspaces of the Cayley space (sometimes called the *irreducible basis vectors*) are:

$$
\begin{aligned}
\chi_1 &= \boxed{\begin{smallmatrix}a&b\\d&c\end{smallmatrix}} + \boxed{\begin{smallmatrix}d&c\\a&b\end{smallmatrix}} + \boxed{\begin{smallmatrix}b&a\\c&d\end{smallmatrix}} + \boxed{\begin{smallmatrix}c&d\\b&a\end{smallmatrix}} \\
\chi_2 &= \boxed{\begin{smallmatrix}a&b\\d&c\end{smallmatrix}} - \boxed{\begin{smallmatrix}d&c\\a&b\end{smallmatrix}} + \boxed{\begin{smallmatrix}b&a\\c&d\end{smallmatrix}} - \boxed{\begin{smallmatrix}c&d\\b&a\end{smallmatrix}} \\
\chi_3 &= \boxed{\begin{smallmatrix}a&b\\d&c\end{smallmatrix}} + \boxed{\begin{smallmatrix}d&c\\a&b\end{smallmatrix}} - \boxed{\begin{smallmatrix}b&a\\c&d\end{smallmatrix}} - \boxed{\begin{smallmatrix}c&d\\b&a\end{smallmatrix}} \\
\chi_4 &= \boxed{\begin{smallmatrix}a&b\\d&c\end{smallmatrix}} - \boxed{\begin{smallmatrix}d&c\\a&b\end{smallmatrix}} - \boxed{\begin{smallmatrix}b&a\\c&d\end{smallmatrix}} + \boxed{\begin{smallmatrix}c&d\\b&a\end{smallmatrix}}.
\end{aligned}
$$

Four irrep basis vectors.

The same information can also be expressed in the:

	$\boxed{\begin{smallmatrix}a&b\\d&c\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}d&c\\a&b\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}b&a\\c&d\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}c&d\\b&a\end{smallmatrix}}$
χ_1	1	1	1	1
χ_2	1	−1	1	−1
χ_3	1	1	−1	−1
χ_4	1	−1	−1	1

Character table for $S_2 \times S_2$.

The one-dimensional subspace generated by any one of the vectors is invariant under the group operations. For instance, if we apply σ_v to χ_3, we get:

$$R_{\sigma_v}(\chi_3) = \boxed{\begin{smallmatrix} b & a \\ c & d \end{smallmatrix}} + \boxed{\begin{smallmatrix} c & d \\ b & a \end{smallmatrix}} - \boxed{\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}} - \boxed{\begin{smallmatrix} d & c \\ a & b \end{smallmatrix}} = -\chi_3.$$

These four irrep basis vectors represent the maximally distinct (e.g., mutually orthogonal) eigen-forms that respect the symmetry operations. Each one is a superposition of the original four basis vectors which were not "symmetry-adapted." By combining the original basis vectors in this way, we get the maximally distinct eigen-forms obeying the symmetry group.

We can also use this example to illustrate how the vector space representations, as opposed to the set representations, generate more variety due to the richer base field. Instead of the Cayley space, we consider the *Cayley set* which is just the set G of group operations which we might represent by the set of four configurations obtained from the initial configuration $\boxed{\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}}$, namely $\left\{\boxed{\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}}, \boxed{\begin{smallmatrix} d & c \\ a & b \end{smallmatrix}}, \boxed{\begin{smallmatrix} b & a \\ c & d \end{smallmatrix}}, \boxed{\begin{smallmatrix} c & d \\ b & a \end{smallmatrix}}\right\}$. Each group operation such as R_{σ_v} acts on these four elements as indicated by the row in the multiplication table. Thus we have a set representation of the group whose orbits will be the minimal invariant subsets. But in every Cayley set representation, there is only one orbit since any element can be mapped to any other element by the action of one of the four operators, i.e., the Cayley group action is transitive. Thus the only eigen-form we get from the Cayley set representation of $S_2 \times S_2$ is the minimal invariant subset or orbit $\left\{\boxed{\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}}, \boxed{\begin{smallmatrix} d & c \\ a & b \end{smallmatrix}}, \boxed{\begin{smallmatrix} b & a \\ c & d \end{smallmatrix}}, \boxed{\begin{smallmatrix} c & d \\ b & a \end{smallmatrix}}\right\}$ which corresponds to χ_1 in the Cayley vector space representation. Thus by working in any base field where $+1 \neq -1$, we immediately get a richer set of symmetry-adapted eigen-forms in addition to the always-present χ_1.

A key part in the above derivation was the fact each R_g could be treated as an operator commuting with all the $R_{g'}$ for $g' \in G$ since the group was Abelian. A few extra wrinkles must be added when the group is not Abelian.

Example 6: The smallest non-Abelian group is the symmetric group S_3 on three elements which is isomorphic to D_3, the full symmetry group for the equilateral triangle. Starting with the initial configuration ${}_b\overset{a}{\triangle}_c$, the non-identity symmetries are (where the "bottom side" of the triangle is "painted black" so the triangle turns black when flipped over and where the vertexes are labelled as in the original configuration ${}_b\overset{a}{\triangle}_c$ when defining the flips):

rotation counterclockwise by 120°, $C_3 : {}_b\overset{a}{\triangle}_c \longmapsto {}_a\overset{c}{\triangle}_b$;

rotation counterclockwise by 240°, $C_3^2 : {}_b\overset{a}{\triangle}_c \longmapsto {}_c\overset{b}{\triangle}_a$;

flip on bisector through vertex a, $C_2^{(a)} : {}_b\overset{a}{\triangle}_c \longmapsto {}_c\overset{a}{\blacktriangle}_b$;

flip on bisector through vertex b, $C_2^{(b)} : {}_b\overset{a}{\triangle}_c \longmapsto {}_b\overset{c}{\blacktriangle}_a$;

flip on bisector through vertex c, $C_2^{(c)} : {}_b\overset{a}{\triangle}_c \longmapsto {}_a\overset{b}{\blacktriangle}_c$.

In the following multiplication table, the column on the left has the inverses of the elements in the top row so that the identity will always be on the diagonal. The operations are represented by the configuration resulting from applying the operation to the initial configuration.

$_{2nd}\backslash^{1st}$	I	C_3	C_3^2	$C_2^{(a)}$	$C_2^{(b)}$	$C_2^{(c)}$
I	${}_{b}\overset{a}{\triangle}_{c}$	${}_{a}\overset{c}{\triangle}_{b}$	${}_{c}\overset{b}{\triangle}_{a}$	${}_{c}\overset{a}{\blacktriangle}_{b}$	${}_{b}\overset{c}{\blacktriangle}_{a}$	${}_{a}\overset{b}{\blacktriangle}_{c}$
C_3^2	${}_{c}\overset{b}{\triangle}_{a}$	${}_{b}\overset{a}{\triangle}_{c}$	${}_{a}\overset{c}{\triangle}_{b}$	${}_{b}\overset{c}{\blacktriangle}_{a}$	${}_{a}\overset{b}{\blacktriangle}_{c}$	${}_{c}\overset{a}{\blacktriangle}_{b}$
C_3	${}_{a}\overset{c}{\triangle}_{b}$	${}_{c}\overset{b}{\triangle}_{a}$	${}_{b}\overset{a}{\triangle}_{c}$	${}_{a}\overset{b}{\blacktriangle}_{c}$	${}_{c}\overset{a}{\blacktriangle}_{b}$	${}_{b}\overset{c}{\blacktriangle}_{a}$
$C_2^{(a)}$	${}_{c}\overset{a}{\blacktriangle}_{b}$	${}_{b}\overset{c}{\blacktriangle}_{a}$	${}_{a}\overset{b}{\blacktriangle}_{c}$	${}_{b}\overset{a}{\triangle}_{c}$	${}_{a}\overset{c}{\triangle}_{b}$	${}_{c}\overset{b}{\triangle}_{a}$
$C_2^{(b)}$	${}_{b}\overset{c}{\blacktriangle}_{a}$	${}_{a}\overset{b}{\blacktriangle}_{c}$	${}_{c}\overset{a}{\blacktriangle}_{b}$	${}_{c}\overset{b}{\triangle}_{a}$	${}_{b}\overset{a}{\triangle}_{c}$	${}_{a}\overset{c}{\triangle}_{b}$
$C_2^{(c)}$	${}_{a}\overset{b}{\blacktriangle}_{c}$	${}_{c}\overset{a}{\blacktriangle}_{b}$	${}_{b}\overset{c}{\blacktriangle}_{a}$	${}_{a}\overset{c}{\triangle}_{b}$	${}_{c}\overset{b}{\triangle}_{a}$	${}_{b}\overset{a}{\triangle}_{c}$

Multiplication table for D_3

The table indicates that the group is not Abelian. We form the *Cayley space* $V = \{D_3 \to \mathbb{C}\} \cong \mathbb{C}^6$ with is spanned by the *standard basis* vectors $|g\rangle$ which take the value 1 on $g \in D_3$ and 0 elsewhere. The operators on the Cayley space $R_{g'} : \mathbb{C}^6 \to \mathbb{C}^6$ are defined by $R_{g'} : |g\rangle \longmapsto |g'g\rangle$ which permutes the basis vectors.

In the previous example of the Abelian group $S_2 \times S_2$, the next step was to consider each R_g like an observable that commutes with all the $R_{g'}$ of the representation. That is not possible for a non-Abelian group so we need to construct operators that do commute with all the group operators and with each other. If we took a subset $C \subseteq \{R_g : g \in D_3\}$, then the requirement that the subset "commute" with all the R_g would be $R_g C = C R_g$ or $R_g C R_{g^{-1}} = C$. That means for any $R_h \in C$ that $R_g R_h R_{g^{-1}} \in C$. If we define a binary relation $R_h \sim R_{h'}$ if $R_g R_h R_{g^{-1}} = R_{h'}$ then that *conjugacy* relation is an equivalence relation, and the equivalence classes are the *conjugacy classes*. A conjugacy class can be turned into a single operator by summing the operators in the class. For a conjugacy class $C \subseteq \{R_g : g \in D_3\}$, let $\mathcal{C} = \sum_{R_g \in C} R_g$ be the *class sum operator* of the class.

A little computation shows that D_3 divides up into three conjugacy classes $\{I\}$ (always a class by itself), $\{C_3, C_3^2\}$, and $\left\{C_2^{(a)}, C_2^{(b)}, C_2^{(c)}\right\}$. If we represent the R_g's by their configurations, then the three class sum operators are:

$$\begin{aligned}
\mathcal{C}_1 &= I = {}_{b}\overset{a}{\triangle}_{c} \\
\mathcal{C}_2 &= C_3 + C_3^2 = {}_{a}\overset{c}{\triangle}_{b} + {}_{c}\overset{b}{\triangle}_{a} \\
\mathcal{C}_3 &= C_2^{(a)} + C_2^{(b)} + C_2^{(c)} = {}_{c}\overset{a}{\blacktriangle}_{b} + {}_{b}\overset{c}{\blacktriangle}_{a} + {}_{a}\overset{b}{\blacktriangle}_{c}.
\end{aligned}$$

Class sum operators

Since the effect of conjugation is only to permute the elements in a conjugacy classes and thus commute the terms in the class sum, the class sum operators commute with all the group operations and thus with each other:

$$R_g \mathcal{C}_i = \mathcal{C}_i R_g \text{ and } \mathcal{C}_i \mathcal{C}_j = \mathcal{C}_j \mathcal{C}_i \text{ for all } g \in D_3 \text{ and } i, j = 1, 2, 3.$$

The class sums are also vectors in the Cayley space V (like ${}_{c}\overset{a}{\blacktriangle}_{b} + {}_{b}\overset{c}{\blacktriangle}_{a} + {}_{a}\overset{b}{\blacktriangle}_{c}$) and they are linearly independent since they are sums of disjoint sets of linearly independent basis vectors. The three-dimensional subspace $V_c \subseteq V$ generated by the class sums is the *class space*. In the class space, we are back in the situation of a space generated by a set of vectors which can also be seen as operators acting on the space, and those operators commute with each other. And since they commute with each other, they will determine a set of simultaneous eigenvectors in the usual manner, and those will be the basis vectors for the irreducible subspaces (of the class space V_c) that are the carriers for the irreps.

But the action of the class sum operators on the class sum vectors in V_c is not just a permutation so we need to construct the multiplication table for the class sum operators. For instance, consider the multiplication:

$$\mathcal{C}_2\mathcal{C}_2 = \left(C_3 + C_3^2\right)^2 = C_3^2 + 2C_3C_3^2 + \left(C_3^2\right)^2 = C_3^2 + 2I + C_3 = 2\mathcal{C}_1 + \mathcal{C}_2.$$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
$\mathcal{C}_1$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
$\mathcal{C}_2$	$\mathcal{C}_2$	$2\mathcal{C}_1 + \mathcal{C}_2$	$2\mathcal{C}_3$
$\mathcal{C}_3$	$\mathcal{C}_3$	$2\mathcal{C}_3$	$3\mathcal{C}_1 + 3\mathcal{C}_2$

Multiplication table for class sum operators
(commutativity indicated by symmetry of table)

Then the operation of the class sum operators on V_c can be described by the following matrices:

$$\mathcal{C}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \mathcal{C}_1 \\ \mathcal{C}_2 \\ \mathcal{C}_3 \end{matrix}, \mathcal{C}_2 = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \mathcal{C}_3 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}.$$

All the vectors of the class space V_c are eigenvectors of $\mathcal{C}_1$ with $\lambda = 1$. The eigenspaces, eigenvalues, and generating eigenvectors of $\mathcal{C}_2$ are:

$$\left\{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}\right\} \leftrightarrow \lambda = -1, \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\} \leftrightarrow \lambda = 2.$$

For $\mathcal{C}_3$, we have:

$$\left\{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}\right\} \leftrightarrow \lambda = 0, \left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\} \leftrightarrow \lambda = -3, \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\} \leftrightarrow \lambda = 3.$$

The generating eigenvectors have been expressed so that they are also the simultaneous eigenvectors. But there is no need in this case to take the join of the eigenspace partitions since the eigenspaces of the one operator $\mathcal{C}_3$ are one-dimensional so it forms a CSCO by itself for the class space. The three simultaneous eigenvectors are the basis vectors for the three irreducible subspaces of V_c and they can be arranged as the:

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

Character table for $D_3 \cong S_3$.

For an Abelian group, the conjugacy classes are all singletons, so the procedure followed in that case is a special instance of the general procedure for non-Abelian groups using class sum operators.

Given two representations $R = \{R_g : V \to V\}_{g\in G}$ and $R' = \{R'_g : V' \to V'\}_{g\in G}$ of the same group G, the representations are said to be *equivalent* if there is a non-singular linear transformation $S : V \to V'$ such that for any $g \in G$, $R_g = S^{-1}R'_g S$. The remarkable fact is the three eigen-forms determined by the simultaneous eigenvectors given in the character table are the only inequivalent irreducible representations used in any representation of D_3, not just the Cayley representation. Any representation of D_3 can be expressed as a direct sum (with repetitions) of those three inequivalent irreps.

The class space is a subspace of the whole Cayley space, but the operator $\mathcal{C}_3$, which was a CSCO in the class space, is not a CSCO in the whole space. As an operator on the whole space,

$$\mathcal{C}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} I \\ C_3^2 \\ C_3 \\ C_2^{(a)} \\ C_2^{(b)} \\ C_2^{(c)} \end{matrix}$$

it has the same eigenvalues, and the eigenspaces V_3 and V_{-3} for $\lambda = 3, -3$ are still one-dimensional, but the eigenspace V_0 for $\lambda = 0$ is four-dimensional. That four-dimensional eigenspace is invariant but it is not minimal. We need to find a new "symmetry-adapted" or "irreducible" basis so that when all the group operations R_g are expressed in that basis, then their matrices will be block-diagonal in the same pattern so that the multiplication of those block-diagonal matrices will only multiply within the blocks. Since the multiplications stay within the blocks, the columns for each block will generate an invariant subspace. When the blocks are of minimum size, then the columns will give irreducible subspaces (some of which may be equivalent). The whole space is a direct sum of those irreducible subspaces, and the original representation is said to be a direct sum of the irreps formed by restricting the representation to the irreducible subspaces.

In the case of the Cayley representation of D_3, it is the direct sum of the first two irreps and two copies of the third one. But the expression of the Cayley representation in these terms is far from unique. In many texts, the required symmetry-adapted basis is generated in a rather *ad hoc* manner. But Jin-Quan Chen and his colleagues in the Nanjing School have developed a CSCO method (i.e., partition joins) to systematically find the irreducible basis vectors for the irreducible spaces that works not only for all representations of finite groups but for all compact Lie groups as needed in QM ([6], [7]). "[T]he foundation of the new approach is precisely the theory of the complete set of commuting operators (CSCO) initiated by Dirac..." [7, p. 2] Thus the linearized partition math of the CSCO method extends also to all compact group representations to characterize the maximally definite eigen-alternatives.

In general, to distinguish indefinite elements into more definite elements, we need more compatible partitions so the joins will have more definite blocks. In the lifted version, more compatible partitions mean more commuting operators whose eigenspace partitions can thus be joined with the given partition $\{V_3, V_{-3}, V_0\}$ to find smaller invariant subspaces until we arrive at minimal ones. Applying the CSCO method of the Chen School, there are two sources of new commuting operators. The CSCOs of a subgroup chain $G \supset G(1) \supset ... \supset G(m)$ is one source.

But each group G has an *opposite group* $\bar{G}$ (Chen calls it the *intrinsic group*) and all the Cayley space constructions can be carried out for the opposite group which is anti-isomorphic to G by the mapping $g^{-1} \longleftrightarrow \bar{g}$. The CSCO of the whole opposite group $\bar{G}$ is of no help in cutting down the invariant subspaces, but the CSCOs of the subgroup chain $\bar{G} \supset \bar{G}(1) \supset ... \supset \bar{G}(m)$ form a second source of commuting operators to reduce the invariant subspaces.

In the case at hand, $D_3 \cong S_3$ is the symmetric group on three elements, and there are three copies of the subgroup S_2, i.e., those generated by $C_2^{(a)}$, $C_2^{(b)}$, or $C_2^{(c)}$. Arbitrarily picking $C_2^{(c)}$, its matrix and the matrix for $\bar{C}_2^{(c)}$ as operators on V are their respective CSCOs:

$$C_2^{(c)} = \begin{bmatrix} 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{bmatrix} \text{ and } \bar{C}_2^{(c)} = \begin{bmatrix} 0&0&0&0&0&1\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0 \end{bmatrix}.$$

The three matrices $\mathcal{C}_3$, $C_2^{(c)}$, and $\bar{C}_2^{(c)}$ commute, and the join of the eigenspace partitions is non-degenerate, i.e., all blocks are one-dimensional. The six simultaneous eigenvectors can be characterized by the kets of eigenvalues (as usual). Grouping the two columns together for each $\bar{C}_2^{(c)}$ eigenvalue gives the matrix whose columns are symmetry-adapted basis vectors that will block-diagonalize the matrices of the original group representation into irreducible blocks:

$$A = \begin{bmatrix} 1&1&1&0&0&-1\\ 1&1&-\frac{1}{2}&-1&1&\frac{1}{2}\\ 1&1&-\frac{1}{2}&1&-1&\frac{1}{2}\\ 1&-1&-\frac{1}{2}&-1&-1&-\frac{1}{2}\\ 1&-1&-\frac{1}{2}&1&1&-\frac{1}{2}\\ 1&-1&1&0&0&1 \end{bmatrix}.$$

Transposed, the new basis vectors as rows have the indicated kets of eigenvalues.

	I	C_3^2	C_3	$C_2^{(a)}$	$C_2^{(b)}$	$C_2^{(c)}$
$\|3,1,1\rangle$	1	1	1	1	1	1
$\|3,-1,-1\rangle$	1	1	1	-1	-1	-1
$\|0,1,1\rangle$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1
$\|0,-1,1\rangle$	0	-1	1	-1	1	0
$\|0,1,-1\rangle$	0	1	-1	-1	1	0
$\|0,-1,-1\rangle$	-1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1

Symmetry-adapted basis in rows with eigenvalue labels

The matrix A represents the symmetry-adapted or irreducible basis vectors in the standard basis, and it can be considered as the change-of-basis matrix $A = C_{St \leftarrow SA}$ to convert the symmetry-adapted basis SA to the standard basis St. Then we can use A and its inverse to convert the five non-identity group representation matrices from the standard to the symmetry-adapted basis.

For instance, the rotate-by-120° matrix in the standard basis and in the symmetry-adapted basis are:

$$C_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } \Gamma\left(C_3\right) = C_{SA\leftarrow St} C_3 C_{St\leftarrow SA} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -1 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{3}{4} \\ 0 & 0 & 0 & 0 & -1 & -\frac{1}{2} \end{bmatrix}$$

which has the appropriate block-diagonal structure. Similarly the matrices for $C_2^{(b)}$ are:

$$C_2^{(b)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } \Gamma\left(C_2^{(b)}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{3}{4} \\ 0 & 0 & 0 & 0 & -1 & \frac{1}{2} \end{bmatrix}$$

and similarly for the other operations.

All the six operations are turned into block-diagonal matrices of the same structure so that when the matrices are multiplied, only the corresponding blocks are multiplied. In this manner the Cayley space is partitioned into the direct sum of the irreducible subspaces $V_1 = \{a_1\}$, $V_2 = \{a_2\}$, $V_3 = \{a_3, a_4\}$, and $V_4 = \{a_5, a_6\}$. The irrep obtained by restricting Γ to V_1, i.e., $\Gamma^{(1)}$, is given by the first character χ_1 and by restricting Γ to V_2, i.e., $\Gamma^{(2)}$, is given by χ_2. Restricting Γ to V_3 and V_4 give two equivalent irreps $\Gamma_1^{(3)}$ and $\Gamma_2^{(3)}$ given by χ_3. Then $V = V_1 \oplus V_2 \oplus V_3 \oplus V_4$ and similarly for the representations: $\Gamma = \Gamma^{(1)} \oplus \Gamma^{(2)} \oplus \Gamma_1^{(3)} \oplus \Gamma_2^{(3)}$.

It is easy to get lost in the detailed mathematics of group representation theory and to overlook the basic theme. The theme is that given the symmetry group, the irreps give all the different ways that maximally definite alternatives can be developed consistent with the symmetries. The irreps fill out the symmetry-adapted possibilities. As always, the indefinite is rendered definite by the distinctions made by the joins of partitions collected together in a complete set. In quantum mechanics, the definite eigen-alternatives of the observables are carved out by the joins of vector space partitions of Dirac's CSCOs. We have also seen that in group representation theory, the properties of the eigen-alternatives (e.g., as determined by the irreps of the symmetry group of the Hamiltonian, or as the elementary particles themselves are determined by the irreps of groups in particle physics[39]) are carved out by the joins of the vector space partitions of Chen's CSCOs–which constitute a "systematic theory ... established for the rep group based on Dirac's CSCO (complete set of commuting operators) approach in quantum mechanics." [6, p. 211]

All the development of the *mathematics* of partitions from sets to vector spaces and now to group representations over vector spaces would be there even if the physical world was perfectly classical. In that case, the mathematics of partitions would just describe some alternative reality, some hypothetical world where there was objective definiteness rather than the classical world of definite properties "all the way down." But that view of the physical world was overthrown by the quantum revolution, and we have found throughout it all that the mathematics used by quantum

[39] For a certain symmetry group of particle physics, "an elementary particle 'is' an irreducible unitary representation of the group."[39, p. 149] Thus our partitional approach comports with "the soundness of programs that ground particle properties in the irreducible representations of symmetry transformations..." [22, p. 171]

mechanics to describe the physical world is precisely this linearized mathematics of partitions describing a world of objective indefiniteness.

12 Appendix 2: The Heisenberg indefiniteness principle in "QM" on sets

In Paul Feyerabend's discussion of Heisenberg's uncertainty[40] principle, he asserted that "inherent indefiniteness is a universal and objective property of matter."[17, p. 202] Thus one path to arrive at the notion of "inherent indefiniteness" is to understand that Heisenberg's indefiniteness principle is *not* about the clumsiness of instruments in simultaneously measuring incompatible observables that always have definite values.

Behind Heisenberg's indefiniteness or indeterminacy principle, the basic idea (not the metrical formula) is that a vector space can have quite different bases so that a ket that is a definite state in one basis is an indefinite superposition in another basis. And that basic idea can be well illustrated at the set level by interpreting $\wp(U)$ as a vector space $\mathbb{Z}_2^n$ (where $|U| = n$) which has many bases. In our previous (simplified) treatment of attributes $f : U \to \mathbb{R}$ and $g : U' \to \mathbb{R}$ not using $\mathbb{Z}_2^n$, the attributes were compatible if $U = U'$. Now we can give a more sophisticated treatment of the set case using $\mathbb{Z}_2^n$, but with the similar result that attributes are compatible, i.e., "commute," if and only if there is a common basis set of "simultaneous eigenvectors" on which both attributes can be defined. The lifted version is the same; two observable operators are compatible if and only if there is a basis of simultaneous eigenvectors, and that holds if and only if the operators commute–which is also equivalent to all the projection operators in the two spectral decompositions commuting.

We are given two basis sets $\{\{a\}, \{b\}, ... \mid a, b, ... \in U\}$ and $\{\{a'\}, \{b'\}, ... \mid a', b', ... \in U'\}$ for $\mathbb{Z}_2^n$ such as in the previous example where $n = 3$ and the U'-basis was the three subset $\{a'\} = \{a, b\}$, $\{b'\} = \{b, c\}$, and $\{c'\} = \{a, b, c\}$. Then we have two real-valued set attributes defined on the different bases, $f : U \to \mathbb{R}$ and $g : U' \to \mathbb{R}$, and we want to investigate their compatibility.

The set attributes define set partitions $\{f^{-1}(r)\}$ and $\{g^{-1}(s)\}$ respectively on U and U'. These set partitions on the basis sets define, as usual, vector space partitions $\{\wp(f^{-1}(r))\}$ and $\{\wp(g^{-1}(s))\}$ on $\mathbb{Z}_2^n$. But those vector space partitions cannot in general be obtained as the eigenspace partitions of Hermitian operators on $\mathbb{Z}_2^n$ since the only available eigenvalues are 0 and 1. But any set attribute that is the characteristic function $\chi_S : U \to \{0, 1\} \subseteq \mathbb{R}$ of a subset $S \subseteq U$ can represented by an operator, indeed a projection operator, whose action on $\wp(U) \cong \mathbb{Z}_2^n$ is given by the "projection operator" $S \cap () : \wp(U) \to \wp(U)$, and similarly for U'. The properties of the real-valued attributes f and g can then stated in terms of these projection operators for subset "eigenspaces" $S = f^{-1}(r) \subseteq U$ and $S' = g^{-1}(s) \subseteq U'$.

Consider first the above example and the simple case where the attributes are just characteristic functions $f = \chi_{\{b,c\}} : U \to \{0, 1\} \subseteq \mathbb{R}$ so $f^{-1}(1) = \{b, c\}$ and $g = \chi_{\{a',b'\}} : U' \to \{0, 1\} \subseteq \mathbb{R}$ so $g^{-1}(1) = \{a', b'\}$. The two projection operators are $\{b, c\} \cap () : \wp(U) \to \wp(U)$ and $\{a', b'\} \cap () : \wp(U') \to \wp(U')$. Note that this representation of the projection operators is basis-dependent. For instance, $\{a', b'\} = \{a, c\}$ but the operator $\{a, c\} \cap ()$ operating on $\wp(U)$ is a very different operator than $\{a', b'\} \cap ()$ operating on $\wp(U')$. The following ket table computes the two projection operators and checks if they commute when the results are stated in the U-basis.

[40] Heisenberg's German word was "Unbestimmtheit" which could well be translated as "indefiniteness" or "indeterminacy" rather than "uncertainty."

U	U'	$f \upharpoonright = \{b,c\} \cap ()$	$g \upharpoonright = \{a',b'\} \cap ()$	$g \upharpoonright f \upharpoonright$	$f \upharpoonright g \upharpoonright$
$\{a,b,c\}$	$\{c'\}$	$\{b,c\}$	$\emptyset$	$\{b,c\}$	$\emptyset$
$\{a,b\}$	$\{a'\}$	$\{b\}$	$\{a'\} = \{a,b\}$	$\{a,c\}$	$\{b\}$
$\{b,c\}$	$\{b'\}$	$\{b,c\}$	$\{b'\} = \{b,c\}$	$\{b,c\}$	$\{b,c\}$
$\{a,c\}$	$\{a',b'\}$	$\{c\}$	$\{a',b'\} = \{a,c\}$	$\{a,b\}$	$\{c\}$
$\{a\}$	$\{b',c'\}$	$\emptyset$	$\{b'\} = \{b,c\}$	$\emptyset$	$\{b,c\}$
$\{b\}$	$\{a',b',c'\}$	$\{b\}$	$\{a',b'\} = \{a,c\}$	$\{a,c\}$	$\{c\}$
$\{c\}$	$\{a',c'\}$	$\{c\}$	$\{a'\} = \{a,b\}$	$\{a,b\}$	$\{b\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Non-commutativity of the projections $\{b,c\} \cap ()$ and $\{a',b'\} \cap ()$.

We can move even closer to QM mathematics by using matrices in $\mathbb{Z}_2^n$ to represent the operators. The U-basis vectors $\{a\}$, $\{b\}$, and $\{c\}$ are represented by the respective column vectors:

$$\begin{bmatrix}1\\0\\0\end{bmatrix}_U, \begin{bmatrix}0\\1\\0\end{bmatrix}_U, \text{ and } \begin{bmatrix}0\\0\\1\end{bmatrix}_U$$

where the subscripts indicate the basis. The projection operator $\{b,c\} \cap ()$ would be represented by the matrix whose columns give the result of applying the operator to the basis vectors:

$$\begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}_U$$

$\{b,c\} \cap ()$ projection matrix in U-basis.

In the U'-basis (with the corresponding basis vectors using the U' subscript), the $\{a',b'\} \cap ()$ projection operator is represented by the projection matrix:

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}_{U'}$$

$\{a',b'\} \cap ()$ projection matrix in U'-basis.

These matrices cannot be meaningfully multiplied since they are in different bases but we can convert them into the same basis to see if they commute. Since $\{a'\} = \{a,b\}$, $\{b'\} = \{b,c\}$, and $\{c'\} = \{a,b,c\}$, the conversion matrix $\mathcal{C}_{U \leftarrow U'}$ to convert U'-basis vectors to U-basis vectors is given as usual by the "brackets" such as $\langle \{a\} |_U \{a'\} \rangle = 1$:

$$\mathcal{C}_{U \leftarrow U'} = \begin{bmatrix}\langle \{a\} |_U \{a'\} \rangle & \langle \{a\} |_U \{b'\} \rangle & \langle \{a\} |_U \{c'\} \rangle\\ \langle \{b\} |_U \{a'\} \rangle & \langle \{b\} |_U \{b'\} \rangle & \langle \{b\} |_U \{c'\} \rangle\\ \langle \{c\} |_U \{a'\} \rangle & \langle \{c\} |_U \{b'\} \rangle & \langle \{c\} |_U \{c'\} \rangle\end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\1 & 1 & 1\\0 & 1 & 1\end{bmatrix}_{U \leftarrow U'}.$$

The conversion the other way is given by the inverse matrix (remember mod (2) arithmetic):

$$\mathcal{C}_{U' \leftarrow U} = \begin{bmatrix}0 & 1 & 1\\1 & 1 & 0\\1 & 1 & 1\end{bmatrix}_{U' \leftarrow U} = \mathcal{C}_{U \leftarrow U'}^{-1}$$

which could also be directly seen from the ket table since $\{a\} = \{b', c'\}$, $\{b\} = \{a', b', c'\}$, and $\{c\} = \{a', c'\}$.

The projection matrix for $\{a', b'\} \cap ()$ in the U'-basis can be converted to the U-basis by computing the matrix that starting with any U-basis vector will convert it to the U'-basis, then apply the projection matrix in that U'-basis and then convert the result back to the U-basis:

$$\begin{gathered} \mathcal{C}_{U \leftarrow U'} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{U'} \mathcal{C}_{U' \leftarrow U} \\ = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}_{U \leftarrow U'} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{U'} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}_{U' \leftarrow U} \\ = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{U} \end{gathered}$$

$\{a', b'\} \cap ()$ projection matrix in the U-basis.

Now the two projection operators are represented as projection matrices in the same U-basis so they can be multiplied to see if they commute:

$$\begin{aligned} g \upharpoonright f \upharpoonright () &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{U} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{U} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{U} \\ f \upharpoonright g \upharpoonright () &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{U} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{U} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{U} \end{aligned}$$

so the two projection matrices do not commute, as we previously saw in the table computation.

There is a standard theorem of linear algebra which holds regardless of base field $\mathbb{Z}_2$ or $\mathbb{C}$:

Proposition 2 *For two diagonalizable (i.e., eigenvectors span the space) linear operators on a finite dimensional space: the operators commute if and only if there is a basis of simultaneous eigenvectors [26, p. 177].*

In the above example of non-commuting projection operators, there is no basis of simultaneous eigenvectors (in fact $\{b, c\} = \{b'\}$ is the only common eigenvector).

In the following example of a third U''-basis where $U'' = \{a'', b'', c''\}$ with the set attributes $f = \chi_{\{a,b\}} : U \to \{0,1\}$ and $h = \chi_{\{a'',c''\}} : U'' \to \{0,1\}$, the projections $\{a, b\} \cap ()$ and $\{a'', c''\} \cap ()$ commute as we see from the last two columns.

U	U''	$f\restriction = \{a,b\} \cap ()$	$h\restriction = \{a'',c''\} \cap ()$	$h\restriction f\restriction$	$f\restriction h\restriction$
$\{a,b,c\}$	$\{a'',c''\}$	$\{a,b\}$	$\{a'',c''\} = \{a,b,c\}$	$\{a,b\}$	$\{a,b\}$
$\{a,b\}$	$\{a''\}$	$\{a,b\}$	$\{a''\} = \{a,b\}$	$\{a,b\}$	$\{a,b\}$
$\{b,c\}$	$\{b'',c''\}$	$\{b\}$	$\{c''\} = \{c\}$	$\emptyset$	$\emptyset$
$\{a,c\}$	$\{a'',b'',c''\}$	$\{a\}$	$\{a'',c''\} = \{a,b,c\}$	$\{a,b\}$	$\{a,b\}$
$\{a\}$	$\{a'',b''\}$	$\{a\}$	$\{a''\} = \{a,b\}$	$\{a,b\}$	$\{a,b\}$
$\{b\}$	$\{b''\}$	$\{b\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{c\}$	$\{c''\}$	$\emptyset$	$\{c''\} = \{c\}$	$\emptyset$	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Commuting projection operators $\{a,b\} \cap ()$ and $\{a'',c''\} \cap ()$.

The change of bases matrices are:

$$\mathcal{C}_{U \leftarrow U''} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{U \leftarrow U''} \text{ and } \mathcal{C}_{U'' \leftarrow U} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so the matrix for the operator $h\restriction = \{a'',c''\} \cap ()$ in the U-basis is:

$$\mathcal{C}_{U \leftarrow U''} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{U''} \mathcal{C}_{U'' \leftarrow U}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_U .$$

Hence we can also check the commutativity of the matrices when both are expressed in the U-basis:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the projections commute, by the above proposition, there is a basis of simultaneous eigenvectors. The three kets that are the simultaneous eigenvectors for the two commuting operators are $|1,1\rangle = \{a,b\} = \{a''\}$, $|1,0\rangle = \{b\} = \{b''\}$, and $|0,1\rangle = \{c\} = \{c''\}$ (where the eigenvalue for $f\restriction$ is given first and for $h\restriction$ second in the kets).

Returning to the two basis sets $\{\{a\},\{b\},... \mid a,b,... \in U\}$ and $\{\{a'\},\{b'\},... \mid a',b',... \in U'\}$ for $\mathbb{Z}_2^n$ with two general real-valued set attributes $f : U \to \mathbb{R}$ and $g : U' \to \mathbb{R}$, the attributes cannot be represented as operators on $\mathbb{Z}_2^n$ but each block $f^{-1}(r)$ and $g^{-1}(s)$ can be analyzed using the projection operators $f^{-1}(r) \cap ()$ and $g^{-1}(s) \cap ()$ for those subsets. Thus we say that attributes f and g *"commute"* if all their projection operators $f^{-1}(r) \cap ()$ and $g^{-1}(s) \cap ()$ commute. Then the above proposition about commuting operators can be applied to the commuting operators to yield the result:

set attributes $f : U \to \mathbb{R}$ and $g : U' \to \mathbb{R}$ "commute" if and only if they are *compatible* in the sense that there is a basis set $\{\{a''\}, \{b''\}, ...\}$ for $\mathbb{Z}_2^n$ whose subsets (vectors) are "simultaneous eigenvectors" for all the projection operators–so that f and g can be taken as being defined on the same basis set of n vectors.

This result also justifies our earlier simplification that f and g were defined as compatible if they were defined on the same set $U = U'$.

In this manner, we see how the essential points (but not the metrical aspects) of Heisenberg's indeterminacy principle, i.e., when two observables can or cannot have simultaneous definite values, are evidenced in the model of "quantum mechanics" over $\mathbb{Z}_2$.

13 Appendix 3: "Unitary evolution" and the two-slit experiment in "quantum mechanics" on sets

To illustrate a two-slit experiment in "quantum mechanics" on sets, we need to introduce some "dynamics." In quantum mechanics, the requirement was that the linear transformation had to preserve the degree of indistinctness $\langle\psi|\varphi\rangle$, i.e., that it preserved the inner product. Where two states are fully distinct if $\langle\psi|\varphi\rangle = 0$ and fully indistinct if $\langle\psi|\varphi\rangle = 1$, it is also sufficient to just require that full distinctness and indistinctness be preserved since that would imply orthonormal bases are preserved and that is equivalent to being unitary. In "quantum mechanics" on sets, we have no inner product but the idea of a linear transformation $A : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ preserving distinctness would simply mean being non-singular.[41] Hence our only requirement on the "dynamics" is that the change-of-state matrix is non-singular (so states are not merged).

Consider the dynamics given in terms of the U-basis where: $\{a\} \to \{a,b\}$; $\{b\} \to \{a,b,c\}$; and $\{c\} \to \{b,c\}$ in one time period. This is represented by the non-singular one-period change of state matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The seven nonzero vectors in the vector space are divided by this "dynamics" into a 4 -orbit: $\{a\} \to \{a,b\} \to \{c\} \to \{b,c\} \to \{a\}$, a 2-orbit: $\{b\} \to \{a,b,c\} \to \{b\}$, and a 1-orbit: $\{a,c\} \to \{a,c\}$.

If we take the U-basis vectors as "vertical position" eigenstates, we can device a "quantum mechanics" version of the "two-slit experiment" which models "all of the mystery of quantum mechanics" [18, p. 130]. Taking a, b, and c as three vertical positions, we have a vertical diaphragm with slits at a and c. Then there is a screen or wall to the right of the slits so that a "particle" will travel from the diaphragm to the wall in one time period according to the A-dynamics.

[41] Moreover, it might be noted that since the "brackets" are basis-dependent, the condition analogous to preserving inner product would be $\langle S|_U T\rangle = \langle A(S)|_{A(U)} A(T)\rangle$ where $A(U) = U'$ is defined by $A(\{u\}) = \{u'\}$. When $A : \mathbb{Z}_2^{|U|} \to \mathbb{Z}_2^{|U|}$ is a linear isomorphism (i.e., non-singular), then the image $A(U)$ of the U-basis is a basis, i.e., the U'-basis, and the "bracket-preserving" condition holds since $|S \cap T| = |A(S) \cap A(T)|$ for $A(S), A(T) \subseteq A(U) = U'$.

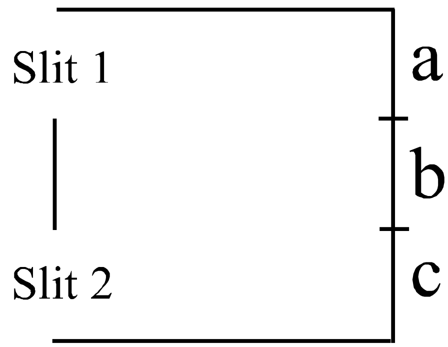


Figure 14: Two-slit setup

We start with or "prepare" the state of a particle being at the slits in the indefinite position state $\{a,c\}$. Then there are two cases.

First case of distinctions at slits: The first case is where we measure the U-state at the slits and then let the resultant position eigenstate evolve by the A-dynamics to hit the wall at the right where the position is measured again. The probability that the particle is at slit 1 or at slit 2 is:

$$\Pr(\{a\} \text{ at slits} \mid \{a,c\} \text{ at slits}) = \frac{\langle \{a\}|_U\{a,c\}\rangle^2}{\|\{a,c\}\|_U^2} = \frac{|\{a\}\cap\{a,c\}|}{|\{a,c\}|} = \frac{1}{2};$$
$$\Pr(\{c\} \text{ at slits} \mid \{a,c\} \text{ at slits}) = \frac{\langle \{c\}|_U\{a,c\}\rangle^2}{\|\{a,c\}\|_U^2} = \frac{|\{c\}\cap\{a,c\}|}{|\{a,c\}|} = \frac{1}{2}.$$

If the particle was at slit 1, i.e., was in eigenstate $\{a\}$, then it evolves in one time period by the A-dynamics to $\{a,b\}$ where the position measurements yield the probabilities of being at a or at b as:

$$\Pr(\{a\} \text{ at wall} \mid \{a,b\} \text{ at wall}) = \frac{\langle \{a\}|_U\{a,b\}\rangle^2}{\|\{a,b\}\|_U^2} = \frac{|\{a\}\cap\{a,b\}|}{|\{a,b\}|} = \frac{1}{2}$$
$$\Pr(\{b\} \text{ at wall} \mid \{a,b\} \text{ at wall}) = \frac{\langle \{b\}|_U\{a,b\}\rangle^2}{\|\{a,b\}\|_U^2} = \frac{|\{b\}\cap\{a,b\}|}{|\{a,b\}|} = \frac{1}{2}.$$

If on the other hand the particle was found in the first measurement to be at slit 2, i.e., was in eigenstate $\{c\}$, then it evolved in one time period by the A-dynamics to $\{b,c\}$ where the position measurements yield the probabilities of being at b or at c as:

$$\Pr(\{b\} \text{ at wall} \mid \{b,c\} \text{ at wall}) = \frac{|\{b\}\cap\{b,c\}|}{|\{b,c\}|} = \frac{1}{2}$$
$$\Pr(\{c\} \text{ at wall} \mid \{b,c\} \text{ at wall}) = \frac{|\{c\}\cap\{b,c\}|}{|\{b,c\}|} = \frac{1}{2}.$$

Hence we can use the laws of probability theory to compute the probabilities of the particle being measured at the three positions on the wall at the right if it starts at the slits in the superposition state $\{a,c\}$ *and* the measurements were made at the slits:

$$\Pr(\{a\} \text{ at wall} \mid \{a,c\} \text{ at slits}) = \tfrac{1}{2}\tfrac{1}{2} = \tfrac{1}{4};$$
$$\Pr(\{b\} \text{ at wall} \mid \{a,c\} \text{ at slits}) = \tfrac{1}{2}\tfrac{1}{2} + \tfrac{1}{2}\tfrac{1}{2} = \tfrac{1}{2};$$
$$\Pr(\{c\} \text{ at wall} \mid \{a,c\} \text{ at slits}) = \tfrac{1}{2}\tfrac{1}{2} = \tfrac{1}{4}.$$

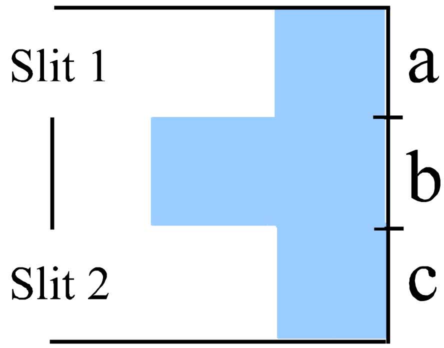


Figure 15: Final probability distribution with measurements at slits

Second case of no distinctions at slits: The second case is when no measurements are made at the slits and then the superposition state $\{a,c\}$ evolves by the A-dynamics to $\{a,b\}+\langle b,c\rangle=\{a,c\}$ where the superposition at $\{b\}$ cancels out. Then the final probabilities will just be probabilities of finding $\{a\}$, $\{b\}$, or $\{c\}$ when the measurement is made only at the wall on the right is:

$$\Pr(\{a\} \text{ at wall} \mid \{a,c\} \text{ at slits}) = \Pr(\{a\} \mid \{a,c\}) = \frac{|\{a\}\cap\{a,c\}|}{|\{a,c\}|} = \frac{1}{2};$$
$$\Pr(\{b\} \text{ at wall} \mid \{a,c\} \text{ at slits}) = \Pr(\{b\} \mid \{a,c\}) = \frac{|\{b\}\cap\{a,c\}|}{|\{a,c\}|} = 0;$$
$$\Pr(\{c\} \text{ at wall} \mid \{a,c\} \text{ at slits}) = \Pr(\{c\} \mid \{a,c\}) = \frac{|\{c\}\cap\{a,c\}|}{|\{a,c\}|} = \frac{1}{2}.$$

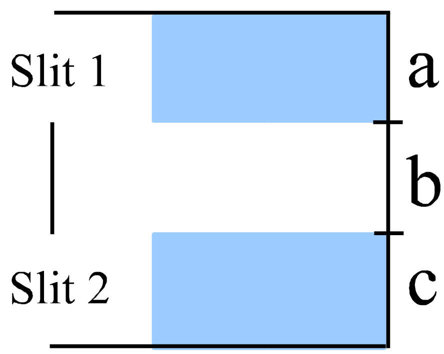


Figure 16: Final probability distribution with no measurement at slits

Since no "collapse" took place at the slits due to no distinctions being made there, the indistinct element $\{a,c\}$ evolved (rather than one or the other of the distinct elements $\{a\}$ or $\{c\}$). The action of A is the same on $\{a\}$ and $\{c\}$ as when they evolve separately since A is a linear operator but the two results are now added together *as part of the evolution*. This allows the "interference" of the two results and thus the cancellation of the $\{b\}$ term in $\{a,b\}+\langle b,c\rangle=\{a,c\}$. The addition is, of course, mod 2 (where $-1=+1$) so, in "wave language," the two "wave crests" that add at the location $\{b\}$ cancel out. When this indistinct element $\{a,c\}$ "hits the wall" on the right, there is an equal probability of that distinction yielding either of those eigenstates.

14 Appendix 4: Bell Theorem in "quantum mechanics" on sets

A simple version of a Bell inequality can be derived in the case of $\mathbb{Z}_2^2$ with three bases $U = \{a, b\}$, $U' = \{a', b'\}$, and $U'' = \{a'', b''\}$, and where the kets are:

kets	U-basis	U'-basis	U''-basis
$\|1\rangle$	$\{a, b\}$	$\{a'\}$	$\{a''\}$
$\|2\rangle$	$\{b\}$	$\{b'\}$	$\{a'', b''\}$
$\|3\rangle$	$\{a\}$	$\{a', b'\}$	$\{b''\}$
$\|4\rangle$	$\emptyset$	$\emptyset$	$\emptyset$

Ket table for $\wp(U) \cong \wp(U') \cong \wp(U'') \cong \mathbb{Z}_2^2$.

Attributes defined on the three universe sets U, U', and U'', such as say $\chi_{\{a\}}$, $\chi_{\{b'\}}$, and $\chi_{\{a''\}}$, are incompatible as can be seen in several ways. For instance the set partitions defined on U and U', namely $\{\{a\}, \{b\}\}$ and $\{\{a'\}, \{b'\}\}$, cannot be obtained as two different ways to partition the same set since $\{a\} = \{a', b'\}$ and $\{a'\} = \{a, b\}$, i.e., an "eigenstate" in one basis is a superposition in the other. The same holds in the other pairwise comparison of U and U'' and of U' and U''.

A more technical way to show incompatibility is to exploit the vector space structure of $\mathbb{Z}_2^2$ and to see if the projection matrices for $\{a\} \cap ()$ and $\{b'\} \cap ()$ commute. The basis conversion matrices between the U-basis and U'-basis are:

$$\mathcal{C}_{U \leftarrow U'} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } \mathcal{C}_{U' \leftarrow U} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The projection matrix for $\{a\} \cap ()$ in the U-basis is, of course, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and the projection matrix for $\{b'\} \cap ()$ in the U'-basis is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Converting the latter to the U-basis to check commutativity gives:

$$\begin{aligned} [\{b'\} \cap ()]_U &= \mathcal{C}_{U \leftarrow U'} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathcal{C}_{U' \leftarrow U} \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

Hence the commutativity check is:

$$\begin{aligned} [\{a\} \cap ()]_U [\{b'\} \cap ()]_U &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \\ [\{b'\} \cap ()]_U [\{a\} \cap ()]_U &= \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

so the two operators for the "observables" $\chi_{\{a\}}$ and $\chi_{\{b'\}}$ do not commute. In a similar manner, it is seen that the three "observables" are mutually incompatible.

Given a ket in $\mathbb{Z}_2^2 \cong \wp(U) \cong \wp(U') \cong \wp(U'')$, and using the usual equiprobability assumption on sets, the probabilities of getting the different outcomes for the various "observables" in the different given states are given in the following table.

Given state \ Outcome of test	a	b	a'	b'	a''	b''
$\{a,b\} = \{a'\} = \{a''\}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	1	0
$\{b\} = \{b'\} = \{a'',b''\}$	0	1	0	1	$\frac{1}{2}$	$\frac{1}{2}$
$\{a\} = \{a',b'\} = \{b''\}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	1

State-outcome table.

The delift of the tensor product of vector spaces is the Cartesian or direct product of sets, and the delift of the vectors in the tensor product are the subsets of direct product of sets (as seen in the above treatment of entanglement in "quantum mechanics" on sets). Thus in the U-basis, the basis elements are the elements of $U \times U$ and the "vectors" are all the subsets in $\wp(U \times U)$. But we could obtain the same "space" as $\wp(U' \times U')$ and $\wp(U'' \times U'')$, and we can construct a ket table where each row is a ket expressed in the different bases. And these calculations in terms of sets could also be carried out in terms of vector spaces over $\mathbb{Z}_2$ where the rows of the ket table are the kets in the tensor product:

$$\mathbb{Z}_2^2 \otimes \mathbb{Z}_2^2 \cong \wp(U \times U) \cong \wp(U' \times U') \cong \wp(U'' \times U'').$$

Since $\{a\} = \{a',b'\} = \{b''\}$ and $\{b\} = \{b'\} = \{a'',b''\}$, the subset $\{a\} \times \{b\} = \{(a,b)\} \subseteq U \times U$ is expressed in the $U' \times U'$-basis as $\{a',b'\} \times \{b'\} = \{(a',b'),(b',b')\}$, and in the $U'' \times U''$-basis it is $\{b''\} \times \{a'',b''\} = \{(b'',a''),(b'',b'')\}$. Hence one row in the ket table has:

$$\{(a,b)\} = \{(a',b'),(b',b')\} = \{(b'',a''),(b'',b'')\}.$$

Since the full ket table has 16 rows, we will just give a partial table that suffices for our calculations.

$U \times U$	$U' \times U'$	$U'' \times U''$
$\{(a,a)\}$	$\{(a',a'),(a',b'),(b',a'),(b',b')\}$	$\{(b'',b'')\}$
$\{(a,b)\}$	$\{(a',b'),(b',b')\}$	$\{(b'',a''),(b'',b'')\}$
$\{(b,a)\}$	$\{(b',a'),(b',b')\}$	$\{(a'',b''),(b'',b'')\}$
$\{b,b\}$	$\{(b',b')\}$	$\{(a'',a''),(a'',b''),(b'',a''),(b'',b'')\}$
$\{(a,a),(a,b)\}$	$\{(a',a'),(b',a')\}$	$\{(b'',a'')\}$
$\{(a,a),(b,a)\}$	$\{(a',a'),(a',b')\}$	$\{(a'',b'')\}$
$\{(a,a),(b,b)\}$	$\{(a',a'),(a',b'),(b',a')\}$	$\{(a'',a''),(a'',b''),(b'',a'')\}$
$\{(a,b),(b,a)\}$	$\{(a',b'),(b',a')\}$	$\{(a'',b''),(b'',a'')\}$

Partial ket table for $\wp(U \times U) \cong \wp(U' \times U') \cong \wp(U'' \times U'')$

As before, we can classify each "vector" or subset as "separated" or "entangled" and we can furthermore see how that is independent of the basis. For instance $\{(a,a),(a,b)\}$ is "separated" since:

$$\{(a,a),(a,b)\} = \{a\} \times \{a,b\} = \{(a',a'),(b',a')\} = \{a',b'\} \times \{a'\} = \{(b'',a'')\} = \{b''\} \times \{a''\}.$$

An example of an "entangled state" is:

$$\{(a,a),(b,b)\} = \{(a',a'),(a',b'),(b',a')\} = \{(a'',a''),(a'',b''),(b'',a'')\}.$$

Taking this "entangled state" as the initial "state," there is a probability distribution on $U \times U' \times U''$ where $\Pr(a, a', a'')$ (for instance) is defined as the probability of getting the result $\{a\}$ if a U-measurement is performed on the left-hand system, and if instead a U'-measurement is performed on the left-hand system then $\{a'\}$ is obtained, and if instead a U''-measurement is performed on the left-hand system then $\{a''\}$ is obtained. Thus we would have $\Pr(a, a', a'') = \frac{1}{2}\frac{2}{3}\frac{2}{3} = \frac{2}{9}$. In this way the probability distribution $\Pr(x, y, z)$ is defined on $U \times U' \times U''$.

A Bell inequality can be obtained from this joint probability distribution over the outcomes $U \times U' \times U''$ of measuring these three incompatible attributes [10]. Consider the following marginals:

$$\begin{aligned}
\Pr\left(a, a'\right) &= \Pr\left(a, a', a''\right) + \Pr\left(a, a', b''\right) \checkmark \\
\Pr\left(b', b''\right) &= \Pr\left(a, b', b''\right) \checkmark + \Pr\left(b, b', b''\right) \\
\Pr\left(a, b''\right) &= \Pr\left(a, a', b''\right) \checkmark + \Pr\left(a, b', b''\right) \checkmark .
\end{aligned}$$

The two terms in the last marginal are each contained in one of the two previous marginals (as indicated by the check marks) and all the probabilities are non-negative, so we have the following inequality:

$$\Pr(a, a') + \Pr(b', b'') \geq \Pr(a, b'')$$

Bell inequality.

All this has to do with measurements on the left-hand system. But there is an alternative interpretation to the probabilities $\Pr(x, y)$, $\Pr(y, z)$, and $\Pr(x, z)$ *if* we assume that the outcome of a measurement on the right-hand system is *independent* of the outcome of the same measurement on the left-hand system. Then $\Pr(a, a')$ is the probability of a U-measurement on the left-hand system giving $\{a\}$ and then a U'-measurement on the right-hand system giving $\{a'\}$, and so forth. Under that *independence assumption* and for this initially prepared "Bell state" (which is left-right symmetrical in each basis),

$$\{(a, a), (b, b)\} = \{(a', a'), (a', b'), (b', a')\} = \{(a'', a''), (a'', b''), (b'', a'')\},$$

the probabilities would be the same.[42] That is, under that assumption, the probabilities, $\Pr(a) = \frac{1}{2} = \Pr(b)$, $\Pr(a') = \frac{2}{3} = \Pr(a'')$, and $\Pr(b') = \frac{1}{3} = \Pr(b'')$ are the same regardless of whether we are measuring the left-hand or right-hand system of that composite state. Hence the above Bell inequality would still hold. But we can use "quantum mechanics" on sets to compute the probabilities for those different measurements on the two systems to see if the independence assumption is compatible with "QM" over $\mathbb{Z}_2$.

To compute $\Pr(a, a')$, we first measure the left-hand component in the U-basis. Since $\{(a, a), (b, b)\}$ is the given state, and (a, a) and (b, b) are equiprobable, the probability of getting $\{a\}$ (i.e., the "eigenvalue" 1 for the "observable $\chi_{\{a\}}$) is $\frac{1}{2}$. But the right-hand system is then in the state $\{a\}$ and the probability of getting $\{a'\}$ (i.e., "eigenvalue" 0 for the "observable" $\chi_{\{b'\}}$) is $\frac{1}{2}$ (as seen in the state-outcome table). Thus the probability is $\Pr(a, a') = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$.

To compute $\Pr(b', b'')$, we first perform a U'-basis "measurement" on the left-hand component of the given state $\{(a, a), (b, b)\} = \{(a', a'), (a', b'), (b', a')\}$, and we see that the probability of

[42] The same holds for the other "Bell state": $\{(a, b), (b, a)\}$.

getting $\{b'\}$ is $\frac{1}{3}$. Then the right-hand system is in the state $\{a'\}$ and the probability of getting $\{b''\}$ in a U''-basis "measurement" of the right-hand system in the state $\{a'\}$ is 0 (as seen from the state-outcome table). Hence the probability is $\Pr(b', b'') = 0$.

Finally we compute $\Pr(a, b'')$ by first making a U-measurement on the left-hand component of the given state $\{(a,a),(b,b)\}$ and get the result $\{a\}$ with probability $\frac{1}{2}$. Then the state of the second system is $\{a\}$ so a U''-measurement will give the $\{b''\}$ result with probability 1 so the probability is $\Pr(a, b'') = \frac{1}{2}$.

Then we plug the probabilities into the Bell inequality:

$$\Pr(a, a') + \Pr(b', b'') \geq \Pr(a, b'')$$
$$\frac{1}{4} + 0 \not\geq \frac{1}{2}$$
Violation of Bell inequality.

The violation of the Bell inequality shows that the independence assumption about the measurement outcomes on the left-hand and right-hand systems is incompatible with "QM" over $\mathbb{Z}_2$ so the effects of the "QM" over $\mathbb{Z}_2$ measurements are "nonlocal."

15 Appendix 5: The "measurement problem" in "QM" on sets

The whole lifting and delifting program corroborates that the logic of partitions is the logic for the reality described by quantum mechanics. To better understand the distinction between von Neumann's type 1 and type 2 processes, i.e., the measurement problem, we consider the delifted version of the problem in "QM" over $\mathbb{Z}_2$.

In "QM" over $\mathbb{Z}_2$, the brackets $\langle T|_U S\rangle = |T \cap S|$ for $T, S \subseteq U$ are basis-dependent. Two abstract vectors or kets $|T\rangle$ and $|S\rangle$ in $\mathbb{Z}_2^n$ (where $|U| = n$) would have to both be expressed in the U-basis to apply the U-bracket $\langle \_|_U \_\rangle$.[43] The set version of a type 2 process would be a linear operator $A : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ that preserves distinctions, i.e., that is non-singular [so that A can be viewed as a change of basis from U to $U' = A(U)$]. This preserves the brackets (changing the basis accordingly), i.e., for any $T, S \subseteq U$,

$$\langle T|_U S\rangle = |T \cap S| = |A(T) \cap A(S)| = \left\langle A(T)\,|_{A(U)} A(S)\right\rangle = \langle A(T)\,|_{U'} A(S)\rangle$$

where $A(T)$ and $A(S)$ are represented in the U'-basis as subsets of $U' = A(U)$. Thus in "QM" over $\mathbb{Z}_2$, the non-singular linear operators $A : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ play the role of the bracket-preserving unitary linear operators in quantum mechanics as describing the type 2 processes.

In the historical development of quantum mechanics out of classical mechanics, the deterministic evolution of the type 2 processes, i.e., the unitary transformations, was seen as the standard case analogous to the classical laws of mechanics, so the type 1 probabilistic transitions were seen as anomalous "quantum jumps" to be explained away if possible. But if, as we have argued, the physical reality described by quantum mechanics is described at the abstract level of logic by the logic of partitions and "QM" over $\mathbb{Z}_2$, then both types of processes are needed and neither should be reducible to the other in QM (or in some modified QM).

[43] Recall that there is no (basis-independent) inner product on a vector space over a finite field.

Having gained some confidence in the lifting and delifting programs, we now propose to approach the measurement problem by looking at the differentiation between "type 1" and "type 2" processes in "QM" over $\mathbb{Z}_2$, and then lifting that solution to quantum mechanics. The "type 2" processes in "QM" over $\mathbb{Z}_2$ are the non-singular linear transformations $A : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$, which can be viewed as change-of-basis transformations which do not change the overlaps between subsets (taking the change of basis into account). The fully-distinct elements $\{u\}$ for $u \in U$ transform into the fully-distinct elements $\{u'\}$ for $u' \in U' = A(U)$ of another basis–just as unitary transformations transform one orthonormal basis into another. No new distinctions are made or unmade in the transformations. Superpositions are linearly preserved.

In contrast, the "type 1" processes in "QM" over $\mathbb{Z}_2$ are those that introduce distinctions to break apart superpositions into smaller, i.e., more definite, superpositions or fully-definite "eigenstates" $\{u\}$. In short:

- "type 2" = distinction-preserving; and
- "type 1" = distinction-making.

Since this characterization of the two different types of processes is formulated abstractly in terms of the lattice of partitions (dual to the lattice of subsets), it is reasonable to ask what are the dual two types of processes in the "classical" Boolean lattice of subsets. Recall from the Figure 1 table that *elements* play the dual role to *distinctions*.[44] Hence we have the:

- dual to the type 2 distinction-preserving process is the element-preserving process, i.e., elements just changing their properties but no elements created or destroyed; and
- dual to the type 1 process of creating (or destroying) distinctions, i.e., moving up (or down) the lattice of partitions, would be the creating (or destroying) elements, i.e., moving up (or down) the lattice of subsets.

Elements may change their properties (e.g., position) in a lawful deterministic way, but any creation (or destruction) of elements would be an entirely different type of process, like the dual creation (or destruction) of distinctions in quantum mechanics.

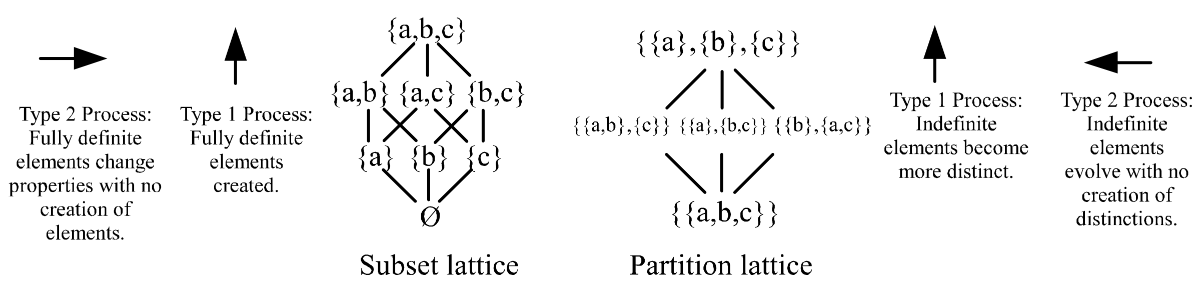


Figure 17: The "Measurement Problem" (i.e., type 1 processes) in the two creation stories.

[44] For an extensive analysis of the categorical duality between elements and distinctions, see [14, § 1.4].

Thus the "measurement problem" in classical mechanics would be giving an account of the type 1 process of the creation of matter *ex nihilo*, i.e., starting with the void (empty set is the bottom of the lattice of subsets)–a process outside the deterministic laws of classical mechanics. An account of element-creation using only the type 2 processes of classical mechanics is just as impossible as an account of distinction-creation (i.e., measurement) using only the type 2 distinction-preserving processes of quantum mechanics.

In terms of the operations of partition logic, the principal distinction-making operation is the join which can be described as the join-action of one partition on a subset, $\pi \vee (C)$, which could be the block of another partition as in the join of two partitions $\pi \vee \sigma$. We might think a subset $S \subseteq U$ as representing a "particle" that is indefinite in some attribute $f : U \to \mathbb{R}$. When the set S is sliced up or "decohered"[45] into the parts $\left\{f^{-1}(r_1) \cap S, f^{-1}(r_2) \cap S, ...\right\}$ by an f-measurement, the "particle" is not sliced up; it becomes more definite in its f-attribute and thus it "jumps" to being represented by one of the more definite subsets $f^{-1}(r) \cap S$ where the "jump" has the probability $\Pr(r|S) = \frac{\left\|f^{-1}(r)\cap S\right\|_U^2}{\|S\|_U^2} = \frac{\left|f^{-1}(r)\cap S\right|}{|S|}$.

Lifting that characterization of "type 1" processes to quantum mechanics, the key to solving the measurement problem, i.e., characterizing the type 1 processes, is making the logical notion of a distinction *physical*. What are the events that make *physical distinctions*? But this is not a new question in QM; it is the old question of *distinguishability*. What are those irreversible physical events, like a hit at a detector, that distinguish between alternatives and thus eliminate the superposition and coherence between alternatives in a superposition state? It is those distinction-making events which distinguish between alternatives that are the lift to QM of the distinction-making operations of partition logic and "QM" over $\mathbb{Z}_2$.

This conceptual characterization of the type 1 processes in QM does not automatically identify those physical distinguishing events. Finding those physical distinguishing events is the physical problem to determine, at least in principle, how to distinguish[46] the eigenstates of an observable when given a superposition of those states, i.e., how to measure the observable. But even the conceptual distinction-making characterization of the type 1 processes does rule out those approaches to the measurement problem that try to deny or explain away the type 1 processes.

It might be useful to review some of the literature from the perspective of this conceptual resolution of the measurement problem. The basic problem has been expressed with great clarity and vigor in John S. Bell's famous essay *Against "Measurement"* [1]. One aspect of the problem is the unclear criterion for separating the processes that constitute measurements from the processes of unitary evolution. In much of the literature, the separation hides behind the distinction between "macroscopic" or "classical" on the one hand, and "microscopic" or "quantum" on the other hand–as if there were precise theoretical definitions of those concepts.

The intuitive idea is that "macroscopic" or "classical" does not countenance any superpositions, so that in the interaction between a macroscopic measurement apparatus and the quantum-level superposition state being "measured," the superposition state will be reduced (or "wave-packet" collapsed) in the measurement process. But considering the measurement apparatus as a large quantum system, the interaction can be modeled by a unitary evolution that would seem to leave

[45] Recall that "decoherence" here always means an actual "reduction" of state, not some "for all practical purposes" reduction. [1]

[46] Here, as always, "to distinguish" means to make the physical distinctions that make an objectively indefinite entity more definite, not the removal of subjective ignorance about an already definite state.

the indicator states of the measurement apparatus in a superposition state (like Schrödinger's cat).

Quite aside from the ill-defined nature of the macroscopic-microscopic distinction, a "measurement" can take place by distinguishing events solely at the quantum level. Feynman [20, §3.3] considered a case of a quantum-level measurement apparatus. A neutron is scattering off the nuclei of atoms in a crystal. If the nuclei have no spin, then the amplitude for the neutron to be scattered to some given point would be the superposition of the scattering amplitudes off the various nuclei since there is no distinguishing physical event to distinguish between scattering off one nucleus or another. But if all the nuclei had spin in, say, the down direction while the neutron had spin up, then in the scattering interaction, one of the nuclei might flip its spin which would be the microscopic physical event to distinguish that trajectory. Then the probability of the neutron arriving at the given point with its spin reversed (indicating that a spin flip had occurred) would be the sum of the probabilities (not the amplitudes) for those distinguished trajectories over all the nuclei. In that case, the superposition was reduced (the indefinite became definite) and the nucleus with its spin flipped plays the role of a detector registering a hit. The spin-state of the nuclei served as a quantum-level measuring apparatus to measure which scattering trajectory was taken by the neutron to reach the detector.

The key concept in this treatment of the measurement problem is the notion of distinguishability, and that concept is perfectly clear at the level of logic. The basic concept in partition logic *is* the notion of a distinction, and that lifts to quantum mechanics as the concept of distinguishability, i.e., as the *distinguishing physical events* that make distinctions between alternatives. Of all the quantum theorists, Richard Feynman is perhaps the clearest in singling out the notion of distinguishability in determining whether or not a superposition of alternatives is reduced (so probabilities add) or not (so amplitudes add). The point is that distinguishing physical events do not allow superposition– which was the real work done by the notions of "macroscopic" or "classical" in the conventional textbook treatment of measurement. And the notion of distinguishing events is derived (lifted) from basic concepts at the level of logic; it is not an *ad hoc* notion, like "macroscopic" or "classical," designed just to plug an embarrassing conceptual gap.

In his crystal-scattering example, the distinguishing physical events are the flips in the spin of nuclei in the crystal. But Feynman's warhorse example is always the two-slit experiment. An electron traverses a screen with two slits and eventually registers a hit on a far wall. The electron traverses the two-slit screen in the superposition state of being indefinite between "going through slit 1" and "going through slit 2." If there is no physical event to distinguish between the two alternative trajectories, then an electron will evolve in its indefinite state and show interference effects (in repeated trials).

If, however, there is some way to, in principle, physically distinguish between the two alternative trajectories, then the superposition is reduced and repeated trials will show no interference effects. The key to this type 1 process is the physical realization of a distinction. In the two-slit experiment, one crude way to physically realize a distinction is to cover up one of the slits. A more subtle way to make a physical distinction is to put a light source between the slits so that photons could bounce off a passing electron to register in a detector.

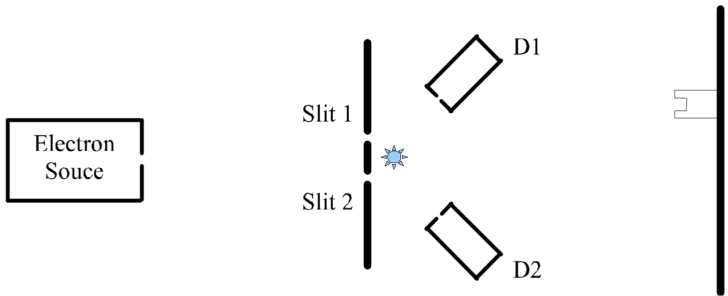


Figure 18: Two slits with detectors

Then the registration of a hit at a detector $D1$ or $D2$ serve as the "non-superposable" physically distinguishing events.

> You must *never add amplitudes for different and distinct final states*. Once the photon is accepted by one of the photon counters, we can always determine which alternative occurred if we want, without any further disturbance to the system. Each alternative has a probability completely independent of the other. ...You do add the amplitudes for the different *indistinguishable* alternatives inside the experiment ... . [20, p. 3-7]

The physics will differ from case to case as to why, say, "going through slit 1" and "going through slit 2" are superposable but hits at detectors $D1$ or $D2$ are not. The important thing here is not the specific way to physically make the distinction, but the concept of *making physical distinctions*–which is the lift to quantum mechanics of the join-action of making distinctions in "QM" over $\mathbb{Z}_2$ and partition logic.

This conceptual solution to the measurement problem can be modeled in "QM" over $\mathbb{Z}_2$. Let $Q = \{a, b\}$ be the quantum system and let $M = \{0, 1, 2\}$ be the measuring apparatus so that $Q \times M$ is the composite system (recall that the tensor product of vector spaces is the lift, via the basis principle, of the direct product of sets). The initial state of the quantum system is the superposition $\{a, b\}$. The state $\{0\}$ is the neutral state of the measuring apparatus, and $\{1\}$ and $\{2\}$ are the pointer states to be correlated respectively with $\{a\}$ and $\{b\}$. The key assumption is that the pointer states are distinguishing states that cannot be superposed. Intuitively we might think of $\{a\}$ and $\{b\}$ as the "going through slit 1" and "going through slit 2" states. The neutral state $\{0\}$ corresponds to the light being off where no measurement is being taken, while the indicator states of $\{1\}$ and $\{2\}$ correspond to the light being on and the detectors $D1$ or $D2$ respectively registering a hit.

In the initial interaction, the composite system is in the superposition state $\{(a, 0), (b, 0)\}$. Then a "type 2" (i.e., non-singular) transformation is applied in the composite system $\mathbb{Z}_2^6$ with the action on the basis as:

$$\begin{array}{c}(a,0) \to (a,1)\\(b,0) \to (b,2)\\(a,1) \to (a,0)\\(b,1) \to (b,1)\\(a,2) \to (a,2)\\(b,2) \to (b,0)\end{array}$$

Action of non-singular transformation.

This gives a "type 2" transformation of the initial state $\{(a,0),(b,0)\}$ to the state $\{(a,1),(b,2)\}$ which correlates the quantum states with the pointer states. But the pointer states $\{1\}$ and $\{2\}$ cannot be superposed. The discrete partition $\mathbf{1}_M = \{\{0\},\{1\},\{2\}\}$ on $M = \{0,1,2\}$ times the indiscrete partition on $Q = \{a,b\}$ gives the product partition

$$\mathbf{0}_Q \times \mathbf{1}_M = \{\{(a,0),(b,0)\},\{(a,1),(b,1)\},\{(a,2),(b,2)\}\}$$

of the composite system which is the least refined partition that still mathematically expresses the assumed *distinguishing nature* of the M-states in the composite system. That distinguishing nature of the M-states is then applied by the join-action of $\mathbf{0}_Q \times \mathbf{1}_M$ on the pure state $\{(a,1),(b,2)\}$ which results in the mixed state $\{\{(a,1)\},\{(b,2)\}\}$.[47] In terms of density matrices, this is the transition that is a "*measurement*":

$$\rho(\{(a,1),(b,2)\}) = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&\frac{1}{2}&0&0&\frac{1}{2}\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&\frac{1}{2}&0&0&\frac{1}{2} \end{bmatrix} \begin{matrix} (a,0)\\ (b,0)\\ (a,1)\\ (b,1)\\ (a,2)\\ (b,2) \end{matrix} \Longrightarrow \hat{\rho} = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&\frac{1}{2}&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&\frac{1}{2} \end{bmatrix}$$

Join-action of Measurement:

$$\{(a,1),(b,2)\} \Longrightarrow \mathbf{0}_Q \times \mathbf{1}_M \vee (\{(a,1),(b,2)\}) = \{\{(a,1)\},\{(b,2)\}\}.$$

Thus with half-half probability, the composite system is in the state $\{(a,1)\}$, i.e., Q is in state $\{a\}$ and M is in state $\{1\}$, or is in the state $\{(b,2)\}$, i.e., Q is in state $\{b\}$ and M is in state $\{2\}$.

Insofar as the quantum system Q is concerned, this is, of course, the same outcomes as described by the "measurement" by an observable $f : Q \to \mathbb{R}$ with $f(a) = 1$ and $f(b) = 2$ so that $f^{-1} = \{\{a\},\{b\}\} = \mathbf{1}_Q$:

$$\rho(\{a,b\}) = \begin{bmatrix} \frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2} \end{bmatrix} \Longrightarrow \hat{\rho} = \begin{bmatrix} \frac{1}{2}&0\\ 0&\frac{1}{2} \end{bmatrix}$$

Measurement: join action of $\{a,b\} \Longrightarrow f^{-1} \vee (\{a,b\})$.

The missing part in the conventional treatment of the measurement problem is that last step of the join-action that expresses the effect of the distinguishing physical events (e.g., the spin-flip of a nucleus in the crystal-scattering example or the hit at a detector in the two-slit experiment) that make distinctions between the superposed alternatives. *That* is the type 1 distinction-making process by which an objectively indefinite system becomes more definite in the measured observable.

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