Physicality and Topological Stability in Relativity Theory

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Abstract

A condition for physicality in general relativity based on topological stability is investigated. Stability in this sense is attractive, for it may seem at first to reduce part of the philosophical problem of understanding physicality to a technical one. What counts as stable, however, depends crucially on the *choice* of topology. Some physicists have thus suggested that one should find a canonical topology, a single "right" topology for every inquiry. While certain such choices might be initially motivated, some little-discussed examples of Geroch [1970, 1971] and some propositions of my own show that the main candidates are not appropriate for every purpose. Although one might find a topology that avoids the particular problems I raise, I suggest instead that the search for a canonical topology is misguided. The conclusion is largely methodological: instead of trying to decide what the "right" topology is for all problems, one should let the details of particular problems guide the choice of an appropriate topology.

1 Stability and Genericity in General Relativity

Despite the fundamental role the notion of *physicality* plays in guiding the practice of physicists, there is no consensus on precisely what it is. Part of the difficulty is the concept's apparent vagueness and overloaded meaning: in different contexts, "unphysical" could, for example, refer to underdetermination, gauge-like overdescription, conflict with experiment, or some more intuitive notion of pathology [Norton, 2008, §3.2]. Some cosmologists propose that a relativistic spacetime can be deemed "unphysical" and excluded from relevance in virtue of it instantiating certain properties, like being extendible, non-isotropic, non-globally hyperbolic, or having holes [Manchak, 2011].[1]

In disentangling these various meanings and uses, it will be helpful to be very specific. Accordingly, this paper explores the *stability* of a property of a relativistic spacetime as a

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[1] See also Earman [1995, Ch. 3.4] for a discussion in the context of the cosmic censorship hypothesis.

necessary condition for that property to be physical. In contrast to the above, this condition bears on the physicality of particular properties of a spacetime rather than on the physicality of the spacetime itself.[2] Roughly, a property of an object O of a specified class (like mathematical models, solutions to a differential equation, etc.) is stable when all objects in that class sufficiently similar to O also have that property. One fairly weak way to formalize stability is to put a topology on this class of objects. This determines notions of convergence for sequences and continuity for parameterized families. The topology's system of open neighborhoods then codifies this similarity mathematically, so that a property P of O (or, more generally, of each of a set of objects $\{O_\alpha\}$) is stable just in case there is an open set containing O (resp. $\{O_\alpha\}$), all of whose members also have P. The name "stability" comes from the intuitive picture that the property is preserved under arbitrary (but sufficiently small) perturbations.

There are several ways to motivate the connection between physicality and topological stability. One involves securing inference under idealization. Scientists often use idealized models to represent phenomena of interest for convenience when more referentially accurate models are intractable. In these cases, one would like to infer properties of the target model from properties of the idealized model. Such an inference will be valid when the idealization is not too severe and the property in question is stable. Another related motivation uses an analogy with the epistemology of experimentation. If one perturbs the object O sufficiently slightly and some property P disappears, one may have reason to believe that P was just an artifact of the imperfect mathematical representation of the phenomena it represents, just as a new experimental effect vanishing upon twiddling unimportant knobs on the apparatus gives one reason to believe that the effect was really an artifact of the messy material assumptions built into one's understanding of the experiment.

While these motivations are not unproblematic and certainly deserve further study, for the remainder of this paper I will set them aside to focus on the connection between physicality and stability for properties of relativistic spacetimes. Recall that a *relativistic spacetime* is an ordered pair (M, g_{ab}), where M is a four-dimensional smooth manifold[3] and g_{ab} is a smooth Lorentzian metric on M, whose indices are abstract.[4] Then the collection of objects to topologize consists of the Lorentzian metrics on a fixed manifold M, which I denote $L(M)$.[5] Stephen Hawking has asserted that "the only properties of space-time that are physically significant are those that are stable in some appropriate topology" [Hawking, 1971, p. 395].[6] Similarly, stability can play a role in making generalizations about classes of spacetimes in

[2]As such, physicality is a second-order property. Cf. Fletcher [2012, §3.1,3.3], who considers the second-order property of being a measure-theoretic null set as condition for unphysicality and possible relationships between physicality of properties and physicality of models.

[3]One also requires M to be connected, paracompact, and Hausdorff. While $\dim(M) = 4$ is of the most physical interest, the proofs of the appendix allows for the more general case of $\dim(M) \geq 2$,

[4]That is, the super- and subscripts of tensor fields like g_{ab} label copies of vector spaces in which the fields reside. See, e.g., Malament [2012, §1.4].

[5]One might of course also wish to compare spacetimes whose underlying manifolds are not identical or even homeomorphic. Although Hawking and Ellis [1973, p. 198] state that this can be done, to my knowledge no one has done so in print.

[6]See also Hawking and Ellis [1973, p. 197]. While I do not completely agree with Hawking's stated justification for his proposal (via "the uncertainty principle"), it is nevertheless independently plausible for the foregoing reasons.

the face of counterexamples that one believes to be exceptional or isolated. In such cases, the property of being a counterexample to the generalization would be unstable, hence would be unphysical and ignorable. In this sense, one would like to talk of a property P holding *generically* on a family A of spacetimes when it holds on an open dense (relative to A) subset of A. Consequently, Hawking says that "For physical purposes it is sufficient to prove that a theorem holds generically" [Hawking, 1971, p. 395].

One attraction of Hawking's proposal is that it seems to reduce part of a philosophical question to a technical question: in certain cases, instead of puzzling over the meaning and use of "physicality," one instead may determine on which sets a property of interest is stable; instead of assessing the significance of apparent counterexamples to a theorem, one simply proves that the theorem holds generically. But a problem arises immediately: there are *infinitely many* topologies one can place on $L(M)$, topologies that can differ regarding whether a property is stable or generic on a set.

2 The Open Topologies

How can one decide which topology to use? Perhaps there is in fact a *canonical* topology: a single choice of topology over $L(M)$ that should apply whenever such a topology is needed. Such a position has been suggested by Geroch, who writes, "It is important, I feel, that one settles on one (or possibly two) topologies in which to work rather than discovering a new topology for each new theorem" [Geroch, 1970, p. 269],[7] and more strongly by Lerner [1972] and Lerner and Porter, who advocate for a particular choice: "if one regards all Lorentz metrics on M as being on an equal (mathematical [*sic*] footing, it appears that the only acceptable choice for a topology is the Whitney fine C^k topology" [Lerner and Porter, 1974, p. 1413]. This topology, also called the C^k *open* topology, may be defined as follows. First, let h^{ab} be some (inverse of a) positive definite metric on M, and define the "distance" function between the k^{th} partial derivatives of two Lorentz metrics g_{ab}, g'_{ab} relative to h^{ab} as

$$d(g,g';h,k) = \begin{cases} [h^{ru}h^{sv}(g_{rs}-g'_{rs})(g_{uv}-g'_{uv})]^{1/2}, & k=0, \\ [h^{a_1b_1}\cdots h^{a_kb_k}h^{ru}h^{sv}\nabla_{a_1}\cdots\nabla_{a_k}(g_{rs}-g'_{rs})\nabla_{b_1}\cdots\nabla_{b_k}(g_{uv}-g'_{uv})]^{1/2}, & k>0, \end{cases} \tag{1}$$

where ∇ is the Levi-Civita derivative operator compatible with h_{ab}. I have omitted the abstract indices in the arguments of d since they needlessly clutter the notation, and I will hereafter continue to drop them when they will never be contracted.

A particular choice of positive definite h effectively determines a coordinate system in which $d(g,g';h,k)$ compares the components of the k^{th} order derivatives of g and g' at each point of M. Then the sets of the form

$$B_k(g,\epsilon;h) = \{g' : \sup_M d(g,g';h,0) < \epsilon, \ldots, \sup_M d(g,g';h,k) < \epsilon\} \tag{2}$$

[7] I do not attribute to him outright advocacy, since a careful reading reveals an admixture of methodological pragmatism: "I think it is important . . . to eventually settle on one or possibly two topologies with which to work. Hardly any economy of thought results if there are hundreds of topologies in use" [Geroch, 1971, p. 73]. Moreover, later writings indicate a preference for the methodologically contextualist approach I take in §4: "The topology one chooses in practice depends on what one wants the topology to do" [Geroch, 1985, p. 175–6].

constitute a basis for the C^k open topology, where g ranges over all Lorentz metrics, ϵ ranges over all positive constants, and h ranges over all positive definite (inverse) metrics. One can view these basis elements as generalizations of the ϵ-balls familiar to metric spaces.

But how does one justify the C^k open topology as canonical? For instance, how should one chose the right value of k? One way is to investigate examples of stability about which one has a strong intuition, ruling out available topologies that do not meet them. For example, in discussing a theorem proving the stability of the strong energy condition[8] in the C^2 open topology, Lerner writes,

> It should be pointed out that [this theorem] is *not* true in any of the weaker topologies frequently used ... If we agree that any reasonable topology ... should allow perturbations preserving the existence of non-zero rest mass, we may take this as further evidence in favor of the [open] topologies. [Lerner, 1973, p. 28]

Indeed, it seems that virtually all of the results regarding stability and genericity of global properties of spacetimes have used one of the open topologies. For example, the encyclopedic monograph *Global Lorentzian Geometry*, which has an entire chapter on "stability of [geodesic] completeness and incompleteness," defines only the open topologies for these purposes [Beem et al., 1996, p. 63 & Ch. 7].

However widely accepted, the universal appropriateness of the open topologies has not gone unquestioned. Geroch [1970, 1971] has provided a pair of examples that illustrate some disquietingly surprising features of the C^0 open topology in particular. To see how they work, recall that a topology determines notions of convergence and continuity. Specifically, a sequence of Lorentz metrics $\overset{m}{g} \in L(M)$ converges to a Lorentz metric $g \in L(M)$ just in case the sequence is eventually contained in every open neighborhood of g. Geroch's first example is a sequence that seems like it *should* converge to Minkowski spacetime but in fact does not. Explicitly, the sequence of metrics

$$\overset{m}{g}_{ab} = \left(1 + \frac{1}{m^2 + x^2 + y^2 + z^2}\right)(d_a t)(d_b t) - (d_a x)(d_b x) - (d_a y)(d_b y) - (d_a z)(d_b z) \qquad (3)$$

on $\mathbb{R}^4$, where t, x, y, z are scalar coordinate fields, does not converge as $m \to \infty$ to the Minkowski metric

$$\eta_{ab} = (d_a t)(d_b t) - (d_a x)(d_b x) - (d_a y)(d_b y) - (d_a z)(d_b z), \qquad (4)$$

even though the "bump," remaining centered at the coordinate origin, decreases in amplitude to zero. Now $\overset{m}{g} \to \eta$ in the C^0 open topology if and only if for *every* neighborhood of the form $B_0(\eta, \epsilon; h)$, we have $\overset{m}{g} \in B_0(\eta, \epsilon; h)$ for m sufficiently large. Pick

$$h^{ab} = (1+x^2+y^2+z^2)^2 \left(\frac{\partial}{\partial t}\right)^a \left(\frac{\partial}{\partial t}\right)^b + \left(\frac{\partial}{\partial x}\right)^a \left(\frac{\partial}{\partial x}\right)^b + \left(\frac{\partial}{\partial y}\right)^a \left(\frac{\partial}{\partial y}\right)^b + \left(\frac{\partial}{\partial z}\right)^a \left(\frac{\partial}{\partial z}\right)^b,$$

[8] This is the condition that for any timelike vector ξ^a at any point of M, $\left(T_{ab} - \frac{1}{2} T g_{ab}\right) \xi^a \xi^b \geq 0$, where T_{ab} is the stress-energy tensor and T is its trace. See Hawking and Ellis [1973, p. 95] or Malament [2012, p. 166].

so that

$$d(\eta, \overset{m}{g}; h, 0) = [h^{ac}h^{bd}(\eta_{ab} - \overset{m}{g}_{ab})(\eta_{cd} - \overset{m}{g}_{cd})]^{1/2} = \frac{(1 + x^2 + y^2 + z^2)^2}{m^2 + x^2 + y^2 + z^2}.$$

However, for any choice of m, the supremum of this expression over the *whole* manifold does not exist, so the sequence does not converge to Minkowski spacetime.

Geroch's second example is the one-parameter family

$$\Lambda = \{\lambda g_{ab} : \lambda > 0\}, \tag{5}$$

with a fixed g_{ab} on a non-compact M, which strikingly does not trace out a continuous curve in the C^0 open topology—indeed, it is everywhere discontinuous because its subspace topology is discrete. Now, the family is continuous in the C^0 open topology if and only if for every $\lambda_0 > 0$ and every neighborhood of the form $B_0(\lambda_0 g, \epsilon; h)$, there is a positive open interval $I \ni \lambda_0$ such that $\{\lambda g_{ab} : \lambda \in I\} \subseteq B_0(\lambda_0 g, \epsilon; h)$. So consider an arbitrary $\delta \in \mathbb{R}$. We have that

$$\begin{aligned} d(\lambda_0 g, (\lambda_0 + \delta)g; h, 0) &= [h^{am}h^{bn}(\lambda_0 g_{ab} - (\lambda_0 + \delta)g_{ab})(\lambda_0 g_{mn} - (\lambda_0 + \delta)g_{mn})]^{1/2} \\ &= |\delta|(h^{am}h^{bn}g_{ab}g_{mn})^{1/2}. \end{aligned}$$

Unless $\delta = 0$, one can easily choose h^{ab} so that this quantity has no supremum over the whole manifold. But if $\delta = 0$, then $I = \{\lambda_0\}$ and cannot be open. So the λg_{ab} cannot trace out a continuous curve. The subspace topology $\mathcal{T}_\Lambda$ on Λ induced by the open topology C^0 is given by $\mathcal{T}_\Lambda = \{\Lambda \cap U : U \in C^0\}$. But as we have just shown, there are $U \in C^0$ such that if $\lambda g_{ab} \in U$, then $\lambda' g_{ab} \notin U$ for $\lambda' \neq \lambda$. Thus $\{\lambda g_{ab}\} \in \mathcal{T}_\Lambda$, but the choice of λ was arbitrary, so $\mathcal{T}_\Lambda$ is discrete.

This example is particularly surprising because one can interpret the elements of Λ to be physically equivalent, representing mere changes of units.[9] In fact, one can prove quite general results regarding the conditions under which a sequence converges or a family is continuous in the open topologies. Specifically, the following is sketched by Golubitsky and Guillemin [1973, p. 43–44]:

Proposition 1. *Let $g, \{\overset{n}{g}\}_{n\in\mathbb{N}}$ be Lorentz metrics on a non-compact manifold M. Then $\overset{n}{g} \to g$ in the open C^k topology iff there is a compact $C \subset M$ such that:*

1. *for sufficiently large n, $\overset{n}{g}_{|M-C} = g_{|M-C}$; and*
2. *$\overset{n}{g} \to g$ on C, i.e., considering C as a submanifold, $\overset{n}{g}_{|C} \to g_{|C}$ in the open C^k topology.*

[9] The example also demonstrates that the open topologies, like the other topologies that I will consider, "over-represent" the physically possible Lorentz metrics on M since in general they represent isometric spacetimes through distinct points. One can compensate for this defect somewhat by ensuring one constructs only invariant topologies [Geroch, 1970, p. 281–2]: ones for which the pushforward map induced by any element of the diffeomorphism group of M acts on $L(M)$ as a homeomorphism. Indeed, all of the topologies considered in this paper are invariant in this way.

In other words, a sequence converges in the C^k open topology just in case its elements eventually *equal* the limit point everywhere except at most on a compact set, a criterion of convergence even stronger than uniform! (Virtually all manifolds of physical interest are non-compact.) One can then use this theorem to prove a necessary condition for a family of metrics to be continuous. (See §A.1 for the proof.)

Proposition 2. *Suppose that $L(M)$ is given the C^k open topology, with M non-compact. If $f : \mathbb{R} \to L(M)$ is continuous, then for every $x_0, x_1 \in \mathbb{R}$, there is some compact $C \subset M$ such that $f(x_0)_{|M-C} = f(x_1)_{|M-C}$.*

Thus any pair from a continuous one-parameter family of metrics must always be equal everywhere except at most on a compact set.[10] Intuitively, one might picture the difference between any such pair as a "bump in a rug" that the function f pushes around. Although the bump may be bigger or smaller, wider or narrower, it always has compact support. This is clearly a quite restricted class of continuous families.

Lerner is aware of Geroch's examples and propositions 1 and 2 [Lerner, 1972], but demurs regarding their significance. Regarding the continuity of examples like eq. 5, he writes that "one-parameter families ought not even to [*sic*] be formulated in the context of [the continuous Lorentz metrics], regardless of the topology" [Lerner, 1972, p. 45]. Thus Lerner would insist radically that parameterized families of spacetimes are simply not useful, but he gives no supporting argument why one should not reject the open topologies instead. Regarding the non-convergence of sequences like eq. 3, he advises that

> This is not a serious problem. In the first place, as Geroch [1969] points out, the process of taking limits is not without ambiguity. Secondly, the limiting metric is often one which, when maximally extended, determines a base manifold $M'' \neq M$. Thus *the problem of limits is not well-posed . . . no matter which topology one uses.* [Lerner, 1973, p. 22, fn. 3]

To clarify, Geroch [1969], a version of whose positive proposal I consider in more detail in §3, points out that the common practice of taking limits of spacetimes *defined within a particular coordinate system* is ambiguous because "by changing coordinates, one can usually obtain some quite different spacetime in the limit" [Geroch, 1969, p. 180]. By contrast, the limits we are concerned with here—and the ones Geroch *goes on to define*—are coordinate-independent, so this ambiguity does not arise. With regards to Lerner's second point, one may well object that it is a restriction of the formalism, not a ill-posedness on the part of the limit, that precludes the treatment of cases of topology change in the limit. Thus his conclusion that considerations from limits and continuity of families are irrelevant even in the case of the open topologies seems unsupported. Consequently, Geroch's examples (eqs. 3 and 5) and their generalizations in propositions 1 and 2 should still give one pause in considering the open topologies as canonical.

[10] In fact, one can prove this for families parameterized by any path-connected topological space. See §A.1.

3 Geometric Continuity and the Compact-Open Topologies

Geroch [1969] has proposed a way of interpreting certain limiting relations entirely geometrically through the continuity (smoothness, etc.) of certain fields. Roughly, in the simplest case of a one-parameter family, one constructs a 5-dimensional manifold from the 4-dimensional manifolds of the family "stacked" by their identifying parameter. More precisely, suppose that one is given a family of metrics $\{\overset{t}{g}_{ab}\}_{t\in\mathbb{R}}$ on a fixed manifold M.[11] Let $\mathcal{M}$ be a manifold diffeomorphic to $M\times\mathbb{R}$ and let $\psi^{(t)}: M\to\mathcal{M}$ be a family of embeddings. If we require that the field $\tilde{t}_{|(p,t)} = (\psi^{(t)}_p)_*\bar{t}$ be smooth, where $\bar{t}$ is a constant field on M of value t, then we can define a symmetric field Γ^{ab} on $\mathcal{M}$ with signature $(+,-,-,-,0)$ by stipulating that $(\Gamma^{ab})_{|(p,t)} = (\psi^{(t)}_p)_*(\overset{t}{g}{}^{ab})$, hence $\Gamma^{ab}\nabla_a\tilde{t} = \mathbf{0}$, where ∇ is a fixed derivative operator satisfying $\nabla_b\nabla_a\tilde{t} = \mathbf{0}$ and $\nabla_a\Gamma^{bc} = \mathbf{0}$.[12] Now, for each $p\in M$ the points $\psi^{(t)}(p)$ for all t form a smooth curve and the collection of all such curves for all p form a congruence on $\mathcal{M}$. Thus there is a vector field τ^a on $\mathcal{M}$ tangent to the curves of this congruence satisfying $\tau^a\nabla_a\tilde{t} = 1$. This allows one at last to define a unique symmetric field Γ_{ab} such that $\Gamma_{ab}\tau^a = \mathbf{0}$ and $\Gamma_{ab}\Gamma^{bc} = \delta^c_a - \tau^c\nabla_a\tilde{t}$.[13] With this construction in place, we can say that the family $\overset{t}{g}_{ab}$ on M is *continuous in the geometric sense* when the corresponding field Γ_{ab} is continuous everywhere on $\mathcal{M}$. (Analogous definitions would apply to smoothness, etc.) One can similarly define the limit of a sequence of metrics by embedding the sequence in a one-parameter family. A strong appeal of the proposal is that it uses the natural, widely accepted geometrical formulation of a relativistic spacetime to do the work of choosing the canonical topology.

It turns out that the topology determined by *all* the geometrically C^k families is the well-known C^k compact-open topology,[14] whose basis elements may be written as sets of the form

$$B_k(g,\epsilon;h,C) = \{g' : \sup_C d(g,g';h,0) < \epsilon, \ldots, \sup_C d(g,g';h,k) < \epsilon\}, \tag{6}$$

where g ranges over all Lorentz metrics, ϵ ranges over all positive constants, h ranges over all positive definite (inverse) metrics, and C ranges over all compact subsets of M. In other words:

Proposition 3. *A family of Lorentz metrics $\{\overset{t}{g}\}_{t\in\mathbb{R}}$ is C^k continuous in the geometric sense iff it is continuous in the C^k compact-open topology.*

For a proof, see §A.2.[15] The essential difference between the open and the compact-open

[11] Geroch does not require that the metrics be defined on the same—or even homeomorphic—manifolds, but we can confine attention to that case here.

[12] For those familiar with the geometrized formulation of Newtonian gravitation, $\nabla_a\tilde{t}$ functions much like the temporal metric and Γ^{ab} like the spatial metric, except that the latter is only assumed to be smooth on each $\tilde{t} =$ const. hypersurface. Thanks to Jim Weatherall for emphasizing this to me.

[13] This also parallels the construction of the covariant spatial metric in geometrized Newtonian gravitation. Cf. fn. 12 and Malament [2012, p. 254, Proposition 4.1.12].

[14] In particular, the C^k compact-open topologies are the *final* topologies respectively associated with the C^k geometrically continuous families, the unique topologies on $L(M)$ that make the C^k geometrically continuous families and no others continuous.

[15] In fact, this equivalence holds for base manifolds of any dimension ≥ 2 and families parameterized by any connected, smooth manifold.

topologies is that the former "control" behavior everywhere on the manifold whereas the latter do so only on compact subsets.

Notably, one can show that, unlike with the open topologies, the sequence defined by eq. 3 converges to the Minkowski metric and the family defined by eq. 5 is continuous relative to the C^k compact-open topologies. They are also attractive for having a number of other interesting features. First, they coincide with the topology of C^k compact convergence—that is, a sequence of metrics $\overset{n}{g} \to g$ on M just when it and its partial derivatives to order k (with respect to the Levi-Civita derivative operator compatible with an arbitrary Riemannian metric on M) do so uniformly on each compact $C \subseteq M$ [Munkres, 2000, p. 283, Theorem 46.2].[16] Second, if a sequence of C^k metrics $\overset{n}{g} \to g$, then g is guaranteed to be at least C^k as well [Munkres, 2000, p. 284, Corollary 46.6]. Third, there is a close connection with homotopy. One can show that a family of Lorentz metrics is continuous in the C^k compact-open topology iff they trace out C^k path in $L(M)$. So, in a way, the C^k compact-open topology encodes which Lorentz metrics can be continuously (to order k) deformed into one another.[17]

Like with the open topologies, however, Geroch has criticized the general appropriateness of the compact-open topologies in treating stability, contending that they rule counterintuitively on the sequence of metrics

$$\overset{m}{g}_{ab} = \left(1 + \frac{m}{1+(x-m)^2}\right)(d_a t)(d_b t) - (d_a x)(d_b x) - (d_a y)(d_b y) - (d_a z)(d_b z) \tag{7}$$

on $\mathbb{R}^4$, where t, x, y, z are natural scalar coordinate fields.[18] "The 'bump' in the metrics becomes larger as it recedes to infinity," he writes, but the "sequence *does* approach Minkowski space in the [C^0 compact-open] topology (because the metrics become Minkowskian in every compact set)." However, "[i]ntuitively, we would not think of this sequence as approaching Minkowski space" [Geroch, 1971, p. 71] (or presumably any spacetime at all). Geroch is right that, given any $\epsilon > 0$, one can find a sufficiently large m such that *at some point* of $\mathbb{R}^4$, $\overset{m}{g}$ differs from the Minkowski metric in its first component by more than ϵ. For example, let η be the Minkowski metric on $\mathbb{R}^4$ (eq. 4) and consider any Riemannian (inverse) metric

$$h^{ab} = \alpha\left(\frac{\partial}{\partial t}\right)^a\left(\frac{\partial}{\partial t}\right)^b + \beta\left(\frac{\partial}{\partial x}\right)^a\left(\frac{\partial}{\partial x}\right)^b + \gamma\left(\frac{\partial}{\partial y}\right)^a\left(\frac{\partial}{\partial y}\right)^b + \delta\left(\frac{\partial}{\partial z}\right)^a\left(\frac{\partial}{\partial z}\right)^b,$$

where $\alpha, \beta, \gamma, \delta$ are smooth positive scalar fields, so that

$$d(\eta, \overset{m}{g}; h, 0) = [h^{ac}h^{bd}(\eta_{ab} - \overset{m}{g}_{ab})(\eta_{cd} - \overset{m}{g}_{cd})]^{1/2} = \frac{m|\alpha|}{1+(x-m)^2}. \tag{8}$$

Since α is continuous, it is bounded on any compact $C \subset \mathbb{R}^4$, so for a given such C, there is a sufficiently large m for which eq. 8 becomes as small as one likes, i.e., $\overset{m}{g} \to \eta$. Since

[16] The compact-open topology coincides with the topology of compact convergence on a function space when the range of the functions is a metrizable space [Munkres, 2000, p. 285–6], and the bundle of Lorentz tensor over M, being a manifold, is metrizable.

[17] Equivalently, the family is continuous in the C^k compact-open topology just when the k-jets of the family belong to the same path component.

[18] The formula for the first term is garbled in Geroch [1971, p. 71], but appears without error in Geroch [1970, p. 280].

Geroch suggests the sequence should not converge, he takes the C^0 compact-open topology to be too coarse.

This example is less convincing than his examples for the open topology, however. First note that the appearance of a growing "bump" in the metric that spreads to infinity is relative to the choice of h, which one could easily choose so that the "bump" instead appears to diminish as it spreads. Second, it is instructive to compare eq. 7 with the sequence of Maclaurin expansions of a real function like $\sin(x)$. For any particular finite-order expansion, one can find a sufficiently large x such that the expansion, evaluated at this x, differs from $\sin(x)$ by as much as one wishes. But if one fixes some compact neighborhood of the origin, then the Maclaurin series converges uniformly on that neighborhood. Similarly, the sequence given by eq. 7 converges to Minkowski spacetime because the C^0 compact-open topology corresponds with the topology of compact convergence. Just as the compact convergence of Maclaurin expansions seems perfectly reasonable, it is not clear why the same cannot be said in the case of sequences of Lorentz metrics.[19]

To Geroch's credit, there are other counterintuitive features of the compact-open topologies that bear even more directly on stability. For example, consider Hawking's theorem: the existence of a global time function is equivalent to stable causality, an absence of closed causal curves that is stable in the *open* C^0 topology [Hawking, 1969]. The following propositions, proved in §A.3,[20] illustrate comments of Hawking [1971, p. 396–7] and Hawking and Ellis [1973, p. 198] that the compact-open topologies are not appropriate for the definition of stable causality because their neighborhoods do not control the behavior of the metric outside of their associated compact sets.

Proposition 4. *Chronology violating spacetimes are dense in $L(M)$ in any of the C^k compact-open topologies.*

The significance of this proposition can be seen through the following corollary.

Corollary 1. *No Lorentz metric is stably causal in any of the C^k compact-open topologies on $L(M)$.*[21]

In particular, according to the compact-open topologies, not only does Minkowski spacetime fail to be stably causal, but *not* having closed timelike curves is unphysical! Accordingly, the analog of Hawking's theorem fails. An alternative but equivalent definition of stable causality brings out why: a spacetime (M, g) is stably causal with respect to the C^0 open topology just when there is a metric g' for which there are no closed causal curves and whose light cones everywhere lie outside those of g. By contrast, when stable causality is defined with respect to the C^0 compact-open topology, the light cones of g' need only lie

[19] It seems that Geroch's intuition would require the "difference" between the limit point and the elements of the tail of the converging sequence to eventually become small at *every* point, i.e., he would require some notion of *uniform* convergence. I discuss one possibility for such a topology in §5.

[20] Although propositions 4 and 6 are stated for the case where $\dim(M) = 4$, they are proved for $\dim(M) \geq 3$. Proposition 5 holds for $\dim(M) \geq 2$.

[21] Cf. proposition 5.1 of Manchak [unpublished], which shows that each metric is homotopic to one that violates chronology. As alluded to above, there is a close connection between homotopy and the compact-open topologies: the C^k homotopy classes correspond with the path components of the C^k compact-open topologies.

outside of those of g on a compact subset of M, leaving the rest unconstrained and ripe for the sprouting of closed causal curves.

Now, if (M, g) already contains a closed timelike curve $\gamma : \mathbb{R} \to M$, then one can pick a local basis element $B_k(g, \epsilon; h, C)$ from any compact-open topology so that $\gamma[\mathbb{R}] \subseteq C$ and ϵ is small enough so that all of its members' light cones are sufficiently close to those of g to still evaluate γ as everywhere timelike. Thus in contrast to the absence of stably causal spacetimes, we have that:

Proposition 5. *Every spacetime containing a closed timelike curve does so stably in any of the C^k compact-open topologies.*

In fact, according to the compact-open topologies, almost all spacetimes contain such closed timelike curves, and almost none do not, as the next proposition and its corollary show.

Proposition 6. *Chronology violating spacetimes are generic in $L(M)$ in any of the C^k compact-open topologies.*

Corollary 2. *Chronological spacetimes are nowhere dense in $L(M)$ in any of the C^k compact-open topologies.*

Insofar as we do not take having closed timelike curves to be a property that nearly all spacetimes should share, these results militate against taking any compact-open topology as canonical.

4 Methodological Contextualism

Any canonical topology on $L(M)$ should have the ability to properly distinguish which sequences converge, which families are continuous, and which properties are stable or generic. But as the previous two sections laid out, the two main classes of topologies in the literature fall short of these goals. The open topologies, advocated by Lerner, seem too fine to treat convergence and continuity. The compact-open topologies, naturally suggested through geometric continuity, seem too coarse for stable causality because their neighborhoods control behavior only on compact sets. Of course, that Geroch's examples *do* evince genuine problems for the former can well be challenged, and one may decide to bite the bullets of propositions 1 and 2 or 4, 5, and 6, but this does not completely resolve the issue of how to choose the canonical topology. Any proponent of a canonical topology must decide without being *ad hoc* on which counterintuitive results to accept and are obliged to provide an explanation as to why the intuitive features thereby denied do not have the significance they seemed to.

But reminding oneself of the way these topologies are used suggests that one need not pick any canonical topology at all. Examining the consequences of adopting one topology over another is a part of the process of deciding which topology will be relevant for a given type of problem. Hawking has emphasized as much: "A given property may be stable or generic in some topologies and not in others. Which of these topologies is of physical interest will depend on the nature of the property under consideration" [Hawking, 1971, p. 396]. Indeed, Geroch's later writings (see fn. 7) have indicated the same. If different topologies correspond to different ways one can specify how spacetimes are similar, it is not surprising that *different* topologies would be natural choices for *different* kinds of questions if those questions bear upon different kinds of properties. It thus seems best to accept a kind of methodological contextualism, where the best choice of topology is the one that captures, as best as one can manage, at least the properties *relevant* to the type of question at hand, ones that relevantly similar spacetimes should share. Thus, in contrast to the canonicalist, I would demand that particular choices of topology must be justified as much as one feasibly can.

Fortunately, one will expect that similar questions will tend to demand similar topologies, so the process of justification need not be started afresh each time. In particular, one should arrive at a particular choice of topology through reflective equilibrium, balancing intuitive demands with mathematical implications, as the many examples and propositions of §2–3 did for the open and compact-open topologies. The more one can accumulate these kinds of results, the more there will be relevant data at hand for a particular type of inquiry so that one can make a sharper, better justified conceptual decision regarding which topology to use. Sometimes this will lead one to reject initially promising and intuitive choices, and sometimes it will reinforce them. One need not postulate that this reflective equilibrium lead to a stable limit; even if one has accumulated many results in favor of using a particular topology for some narrow type of inquiry, one should still be open to new facts and connections that will disturb one's equilibrium.[22]

There is a very general way to illustrate how the many demands of various types of inquiries pull in different, sometimes incompatible directions. Recall that the stability of a property depends on the *existence* of a certain open set. Thus it is in a sense *easier* for a property to be stable in a finer topology, since there are more open sets available. In particular, if a property is stable on a certain set in a given topology $\mathcal{T}$, it is stable in *every* topology finer than $\mathcal{T}$. Similarly, recall that the convergence of a sequence depends on certain aspects of *every* open neighborhood of its purported limit point. Thus it is in a sense *easier* for a sequence to converge in a coarser topology, since there are fewer open sets that must fulfill the proper role. In particular, if a sequence converges in a given topology $\mathcal{T}$, it converges in *every* topology coarser than $\mathcal{T}$. Therefore considerations of stability and convergence are in tension with each other: we would expect that some topologies are fine enough for the stability of some properties but too fine for certain sequences to converge, like with the open topology, or vice versa, like with the compact-open topology.

5 New Problems, New Hopes

Methodological contextualism about topologies—at least in the sense of allowing oneself to pick the most appropriate topology for a given application instead of deciding on one in advance—would make all the above worries associated with picking a canonical topology moot. Without a canonical topology, however, stability itself does not directly settle any conceptual questions about physicality. Indeed, the choice of an "intuitively appropriate" topology itself does much of the work, because that choice builds into the formalism those intuitions regarding how different spacetimes should be considered similar.

[22]Cf. the use of reflective equilibrium in moral theorizing [Schroeter, 2004].

More work needs to be done, therefore, characterizing how particular choices of topology may be appropriate for a given kind of question. In the case of stability, one may be able to characterize classes of properties to which particular topologies are (in)sensitive, or the range of topologies in which interesting properties, like stable causality, behave as one might expect. (See the end of the appendix for a discussion of some possibilities for the compact-open topologies.)

Are there topologies in which important theorems hold but which do not suffer the defects of the open and compact-open topologies? Part of the difficulty in answering this question stems from the small variety of topologies used in the literature. Theorems about the stability and genericity of global properties generally use the open topologies (e.g., see Hawking and Ellis [1973, p. 198], Lerner [1973], and Beem et al. [1996, Ch. 7]). Theorems about the stability of Cauchy developments use variants on the coarser compact-open topologies (see Hawking [1971, p. 398–9] and Hawking and Ellis [1973, p. 252–254]). Theorems concerning the convergence of relativistic spacetimes to Newtonian spacetimes (e.g., Malament [1986]) use (implicitly) a point-open topology, which is even coarser.[23]

Now, there is a simple modification to the C^0 open topology that makes the one-parameter families defined by eq. 5 everywhere continuous while, unlike the compact-open topology, still preventing the sequence defined by eq. 7 from converging. Take the basis elements of the C^0 open topology (eq. 2), but restricted only to *bounded pairs* (g, h), ones for which $\sup_M d(g, \lambda g; h, 0) < \infty$ for any positive λ. This prohibits choosing a positive-definite h scaled by a conformal factor that grows too rapidly, eliminating the open neighborhoods of each Lorentzian g that forced the scale-factor families to be everywhere discontinuous. One can show, moreover, that this topology lies between the open and compact-open topologies in coarseness. However the sequence defined by eq. 3 still does not converge to Minkowski spacetime according to this topology, so it still would not rule in the intuitively "right" way according to Geroch.

But if further refinements are found that produce a topology satisfying Geroch's desiderata, might that topology end up being satisfactory for all demands? If I allow for the possibility that the methods available for picking an appropriate topology may single out a unique choice, or perhaps very few, to what extent is methodological contextualism really distinguished from a slightly liberalized canonicalism? The answer is methodological. The two positions are not distinct because of differing ends—whether to use one topology or many—but because of their differing means: what grounds we might have to prefer one topology over another, and whether those grounds need to be articulated. A canonicalist holds that because there are definitive reasons always to choose a single topology (or perhaps very few), there is no reason to say why that choice is appropriate for a given type of inquiry. By contrast, the contextualist takes the relevant reasons to be provided by the type problem at hand, not in advance, and that they should therefore be articulated and reasonably defended. It bears emphasizing that the latter does *not* deny that there can be principled reasons to pick out a certain topology, only that those reasons can ever be given in enough generality to preclude attention to the details of the type of situation at hand. We indeed be may be lucky for the sake of our economy of thought if a few topologies are always

[23]The point-open topologies are defined similarly to the compact-open topologies (eq. 6), but require the suprema be taken over only finitely many points in each basis element instead of over compact sets.

appropriate, but we should not obstruct the development of new ones if they fit particular purposes better.

A Proofs of Propositions

A.1 Continuity and the Open Topologies

Proposition 7. *Let X be any path-connected topological space, and suppose that $L(M)$ is given the C^k open topology, with M non-compact. If $f : X \to L(M)$ is continuous, then for every $x_0, x_1 \in X$, there is some compact $C \subset M$ such that $f(x_0)_{|M-C} = f(x_1)_{|M-C}$.*

Proof. The case where f is a constant function is immediate, so suppose otherwise and pick arbitrary distinct $x_0, x_1 \in X$. By the definition of path-connectedness, there is a continuous function $f' : [0,1] \to X$ such that $f'(0) = x_0$ and $f'(1) = x_1$. Hence the function $f'' = f \circ f' : [0,1] \to L(M)$ is continuous.

I claim that, given any $r \in [0,1]$, there is an open (relative to $[0,1]$) interval $I_r \subseteq [0,1]$, containing r, such that for any $q \in I_r$, there is some compact $C(r,q) \subset M$ for which $f''(r)_{|M-C(r,q)} = f''(q)_{|M-C(r,q)}$. For suppose otherwise, and consider any sequence of intervals $I_r^1 \supset I_r^2 \supset \ldots$ such that $I_r^n \subseteq [0,1]$ for each n and $\bigcap_{n=1}^{\infty} I_r^n = \{r\}$. One can then construct by induction a sequence of metrics that converges to $f''(r)$ but contradicts proposition 1. For the base step, let $I_r^{s_1} = I_r^1$ and note that there is some $q_1 \in I_r^{s_1}$ distinct from r such that $f''(r)_{|M-C} \neq f''(q_1)_{|M-C}$ for any compact $C \subset M$. (One may choose $q_1 \neq r$ because f'' is continuous.) For the inductive step, suppose $I_r^{s_n}$ is given so that there is some $q_n \in I_r^{s_n}$ distinct from r such that $f''(r)_{|M-C} \neq f''(q_n)_{|M-C}$ for any compact $C \subset M$. Then pick some $I_r^{s_{n+1}}$ such that $s_{n+1} > s_n$ and $q_n \notin I_r^{s_{n+1}}$, noting that there is some $q_{n+1} \in I_r^{s_{n+1}}$ distinct from r such that $f''(r)_{|M-C} \neq f''(q_{n+1})_{|M-C}$ for any compact $C \subset M$. The induction is complete, so by construction the sequence $q_n \to r$ as $n \to \infty$, and for each n, $f''(r)_{|M-C} \neq f''(q_n)_{|M-C}$ for any compact $C \subset M$. But because f'' is continuous, it follows that $f''(q_n) \to f''(r)$ as $n \to \infty$ [Munkres, 2000, p. 130, Theorem 21.3], and by proposition 1, this implies in turn that there is a compact $C \subset M$ for which $f''(r)_{|M-C} = f''(q_n)_{|M-C}$ for sufficiently large n, which is a contradiction.

Next, note that the $\{I_r : r \in [0,1]\}$ form an open cover of $[0,1]$ (relative to $[0,1]$). The interval is compact, so by definition there is some finite subcover $\{I_{r_i} : i = 1, \ldots, m\}$, each of whose elements has, for all $q \in I_{r_i}$, an associated compact $C(r_i, q) \subset M$ for which $f''(r_i)_{|M-C(r_i,q)} = f''(q)_{|M-C(r_i,q)}$. One may assume, without loss of generality, that $r_1 < \ldots < r_m$ and that, because the interval is one-dimensional, no point of $[0,1]$ is included in more than two of the I_{r_i}. Thus pick any $q_i \in I_{r_i} \cap I_{r_{i+1}}$ for $i = 1, \ldots, m-1$ and put $q_0 = 0$ and $q_m = 1$. Let $C = \bigcup_{i=1,\ldots,m} C(r_i, q_{i-1}) \cup C(r_i, q_i)$ and observe that

$$\begin{aligned} f(x_0)_{|M-C} &= f''(0)_{|M-C} = f''(q_0)_{|M-C} = f''(r_1)_{|M-C} = f''(q_1)_{|M-C} = \cdots \\ &= f''(q_{m-1})_{|M-C} = f''(r_m)_{|M-C} = f''(q_m)_{|M-C} = f''(1)_{|M-C} = f(x_1)_{|M-C}. \end{aligned}$$

But C is compact and x_0, x_1 were arbitrary, so the proof is complete. □

Proposition 2 then follows as a special case. Note that, if X is instead a space with multiple path components, the same proof goes through for each path component, but then elements of different path components need not be equal on a co-compact set.

A.2 Equivalence of Geometric Continuity and Compact-Open Continuity

To show that the C^k geometric continuity of a family of metrics is equivalent to its continuity in the C^k compact-open topology, it will be helpful to use an alternative characterization of the latter using fiber bundles. In particular, I will use the bundle of Lorentz tensors over M, which I will denote $\hat{L}(M)$, and the k-jet bundle of cross sections of $\hat{L}(M)$, denoted $J^k(M, \hat{L}(M))$. The k-jet of a cross-section $\hat{g}$ at $p \in M$, denoted $j^k\hat{g}_{|p}$, is the equivalence class of sections whose partial derivatives in an arbitrary coordinate system at p are equal to those of $\hat{g}$ up to order k.[24] The C^k compact-open topology then has as a subbasis sets of the form

$$O_k(C, U) = \{g : j^k\hat{g}[C] \subseteq U\}, \tag{9}$$

where C ranges over all compact subsets of M and U ranges over all open sets of the manifold topology of $J^k(M, \hat{L}(M))$. Thus there is a canonical bijection between the C^k compact-open topology on Lorentz metrics given by eq. 6 and the (C^0) compact-open topology on the k-jets of these metrics, considered as cross-sections.

The desired equivalence is essentially a corollary of the following theorem (adapted from Munkres [2000, p. 287, Theorem 46.11]).

Proposition 8. *Let X and Y be topological spaces, and give the set of continuous functions from X to Y, denoted $\mathcal{C}(X, Y)$, the (C^0) compact-open topology. If $f : X \times Z \to Y$ is continuous, then so is the induced function $F : Z \to \mathcal{C}(X, Y)$ defined by the equation $(F(z))(x) = f(x, z)$. The converse holds if X is locally compact*[25] *and Hausdorff.*

In fact, in using proposition 8 one need not even restrict attention to four-dimensional base manifolds or one-parameter families. Any family parameterized by a smooth, connected manifold will do, and the procedure for constructing the manifold analogous to $\mathcal{M}$ is the same, only instead of having a metric of defect one, the defect will be the dimension of the parameterizing manifold.

Proposition 9. *A family of Lorentz metrics $\{\overset{x}{g}\}_{x \in X}$ on M parameterized by a smooth, connected manifold X is C^k continuous in the geometric sense iff it is continuous in the C^k compact-open topology.*

Proof. Let $n = \dim(M)$ and $m = \dim(X)$. Suppose that the family $\overset{x}{g}_{ab}$ is C^k continuous in the geometric sense. Each $(n+m)$-dimensional metric Γ_{ab} corresponds to a cross-section $\hat{\Gamma}$ of a bundle $\Gamma(\mathcal{M})$ of $(n+m)$-dimensional metrics over $\mathcal{M}$, whose partial derivatives to order k are encoded in the k-jet $j^k\hat{\Gamma}$. Then the smooth bundle map $\phi : J^k(\mathcal{M}, \Gamma(\mathcal{M})) \to J^k(M, \hat{L}(M))$ induced by the projection $\pi : \mathcal{M} \to M$ can be composed with $\hat{\Gamma}$ to yield the function

[24] For more on jets and their bundles, see Golubitsky and Guillemin [1973, Ch. 2.2].

[25] A topological space is locally compact when each point has a compact neighborhood.

$f = \phi \circ \hat{\Gamma} : \mathcal{M} \cong M \times X \to J^k(M, L(M))$, which is C^k because $\hat{\Gamma}$ is C^k by hypothesis. Proposition 8 then entails that the map $F : X \to \mathcal{C}(M, J^k(M, \hat{L}(M)))$ defined by $F : x \mapsto j^k\overset{x}{\hat{g}}$ is continuous. But the range of F is just the set of k-jets of cross-sections of $\hat{L}(M)$ with the (C^0) compact-open topology, which is canonically bijective with the C^k compact-open topology on Lorentz metrics.

Conversely, suppose that the family $\overset{x}{g}_{ab}$ is continuous in the C^k compact-open topology, or equivalently, that the map $F : X \to \mathcal{C}(M, J^k(M, \hat{L}(M)))$ defined by $F : x \mapsto j^k\overset{x}{\hat{g}}$ is continuous, where $\mathcal{C}(M, J^k(M, \hat{L}(M)))$ is given the (C^0) compact-open topology. Since M is locally compact and Hausdorff, proposition 8 entails that the map $f : M \times X \to J^k(M, \hat{L}(M))$ is continuous. Thus $(\overset{x}{g}_{ab})_{|p}$ is jointly C^k in x and p.

Let $\psi^{(x)} : M \to \mathcal{M}$ denote the embeddings that yield the (inverse) metric Γ^{ab}, which is C^k when, for any smooth field α_{ab} on $\mathcal{M}$, $\alpha_{ab}\Gamma^{ab}$ is C^k. Now for any $p \in M$ and $x \in X$, $(\alpha_{ab}\Gamma^{ab})_{|\psi^{(x)}(p)} = (\psi^{(x)}_p)^*(\alpha_{ab})\overset{x}{g}^{ab}$; by assumption $(\psi^{(x)}_p)^*(\alpha_{ab})$ is smooth; and $\overset{x}{g}^{ab}$ is C^k because its inverse is. Thus Γ^{ab} is C^k, so Γ_{ab} must be C^k by construction. □

Proposition 3 then follows as a special case.

A.3 Stable Causality and the Compact-Open Topologies

The proof of proposition 4 first requires a lemma allowing one to "interpolate" between any spacetime and a chronology-violating region of Gödel spacetime.[26] This interpolation can be effected outside of any compact set, which ruins any compact-open topology's ability to control it.

Lemma 1. *Let (M, g) and (M', g') be two spacetimes, with $\dim(M) = n$ and $M' = \mathbb{R}^n$. For any embedded submanifold $S \subset M$ and any compact embedded submanifold $R \subset M'$ with $\dim(S) = \dim(R) = n$, there is a spacetime (M, g'') such that $g''_{|M-S} = g_{|M-S}$ and $g'_{|R}$ is isometric to $g''_{|U}$ for some compact $U \subset S$.*

Proof. Pick a chart (V, φ) of M such that $V \subseteq S$ and $\varphi[V]$ is an open ball of radius 4, i.e., $\varphi[V] = B_{\mathbb{R}^n}(\vec{0}, 4) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| < 4\}$, where $\|\cdot\|$ is the Euclidean norm on the coordinates $\vec{x} \in \mathbb{R}^n$. For brevity, define $A_i = \varphi^{-1}[B_{\mathbb{R}^n}(\vec{0}, i)]$ for $i = 1, 2, 3$, and let r be a scalar field on V defined by $r_{|p} = \|\varphi(p)\|$. Finally, define a diffeomorphism $\psi : M' \to V$ such that $\psi[R] \subset A_1$ and put $U = \psi[R]$, which must be compact since ψ is continuous.

Because all Lorentz metrics on $\mathbb{R}^n$ are homotopic [Finkelstein and Misner, 1959], $\psi_*(g')$ is homotopic to $g_{|V}$, considering V as a submanifold. Thus there is some continuous function $f : [0, 1] \to L(V)$ such that $f(0) = \psi_*(g')$ and $f(1) = g_{|V}$. One can then define the continuous Lorentz metric

$$\gamma_{|p} = \begin{cases} g_{|p}, & p \in M - A_3, \\ f(r_{|p} - 2)_{|p}, & p \in A_3 - A_2, \\ [\psi_*(g')]_{|p}, & p \in A_2. \end{cases}$$

In order to produce the desired smooth metric g'', one can convolve γ with an appropriate positive, symmetric mollifier on the region $V - A_1$. In more detail, define $w : \mathbb{R} \to \mathbb{R}$ to be

[26] For more on the properties of Gödel spacetime, see Malament [2012, Ch. 3.1].

the smooth function

$$w(x) = \begin{cases} ce^{-1/(1-x^2)}, & |x| < 1 \\ 0, & |x| \geq 1, \end{cases}$$

where c is a positive constant chosen so that $\int_{\mathbb{R}} w(x)dx = 1$. Further, define $W : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ as the jointly smooth function $W(\vec{x}, \delta) = \delta^{-n} w(\|\vec{x}\|/\delta)$, where $W(\vec{x}, 0) = \lim_{\delta \to 0} W(\vec{x}, \delta)$ is the Dirac delta, the convergence being understood in the distributional sense. Now, one can express γ in terms of its matrix components $\gamma_{\alpha\beta}(\vec{x})$ determined by the chart (V, φ), allowing one to define on $\varphi[V - A_1]$ for some fixed δ the new components

$$\tilde{\gamma}_{\alpha\beta}(\vec{x}) = \int_{\varphi[\mathrm{int}(V - A_1)]} W(\vec{x} - \vec{y}, e\delta c^{-1} w(\|\vec{x}\| - 5/2)) \gamma_{\alpha\beta}(\vec{y}) d\vec{y}, \tag{10}$$

which are smooth on $\varphi[V - A_1]$.[27] Moreover, for sufficiently small δ, the $\tilde{\gamma}_{\alpha\beta}$ approximate the $\gamma_{\alpha\beta}$ arbitrarily well on $V - A_1$.[28] Therefore such $\tilde{\gamma}_{\alpha\beta}$ are the components of a smooth Lorentz metric $\tilde{\gamma}$ on $V - A_1$. Note that, in the integrand of eq. 10 the function W becomes the Dirac delta for $\|\vec{x}\| \geq 7/2$ and $\|\vec{x}\| \leq 3/2$, so on the points of V corresponding to these coordinate regions, $\tilde{\gamma}$ is equal to g and $\psi_*(g')$, respectively. We can define at last

$$g''_{|p} = \begin{cases} g_{|p}, & p \in M - V, \\ \tilde{\gamma}_{|p}, & p \in V - A_1, \\ [\psi_*(g')]_{|p}, & p \in A_1. \end{cases}$$

By construction, $g''_{|M-S} = g_{|M-S}$ since $M - S \subseteq M - V$ and $g''_{|U}$ is isometric to $g'_{|R}$ since $g''_{|A_1} = [\psi_*(g')]_{|A_1}$ and $U = \psi[R] \subset A_1$. □

If one picks (M', g') to be Gödel spacetime and R to be a region containing closed timelike curves, then, following Manchak [unpublished], one can use the construction of g'' above to show that any spacetime (of dimension at least three) is homotopic to one containing closed timelike curves in a compact region. This answers positively some questions posed by Stein [1970, p. 594], although the construction says little about how realistic such spacetimes are. Nevertheless, it is essentially this idea that is used to prove the following.

Proposition 10. *For any manifold M, if $\dim(M) \geq 3$, chronology violating spacetimes are dense in $L(M)$ with any of the C^k compact-open topologies.*

Proof. Any spacetime for which M is compact contains closed timelike curves [Hawking and Ellis, 1973, Prop. 6.4.2, p. 189], so suppose M is non-compact. Let an arbitrary Lorentz metric g on M be given, and note that any neighborhood thereof in the C^k compact-open topology contains a set of the form $B_k(g, \epsilon; h, C)$. Pick $S = M - C$ so that by lemma 1 there is a metric g' such that $g'_{|C} = g_{|C}$ and $g'_{|U}$ is isometric to a chronology violating region of Gödel spacetime for some $U \subset M - C$. Thus $g' \in B_k(g, \epsilon; h, C)$ and violated chronology, but the choice of g and its neighborhood was arbitrary, so the chronology violating spacetimes are by definition dense in $L(M)$. □

[27] Cf. the proof in Oden and Reddy [1976, Theorem 2.6, p. 48–49], beyond which nothing new is needed.

[28] Oden and Reddy [1976, Theorem 2.7, p. 49] show that, for the case where the integrand contains $W(\vec{x} - \vec{y}, \delta)$ with a fixed δ, the analog of eq. 10 would converge to $\gamma_{\alpha\beta}$ as $\delta \to 0$ in $L^p(\varphi[\mathrm{int}(V - A_1)])$-norm. As before (cf. footnote 27), allowing δ to smoothly vary introduces no new complications.

Proposition 4 and its corollary then follow as a special case. The proof of proposition 5 is a straightforward computation:

Proposition 11. *If (M, g) contains a closed timelike curve, then g is stably chronology violating in every compact-open topology.*

Proof. Fix any positive definite metric h_{ab} and note that one can write $g_{ab} = h_{am}\mu^m h_{bn}\mu^m - h_{ab}$ for some smooth vector field μ^a [Hawking and Ellis, 1973, p. 39]. One can thus express that $\gamma : I \to M$ is a closed g-timelike curve with tangent vector ξ^a as the condition that $|h_{ab}\xi^a\mu^b|_{|\gamma[I]} > (h_{ab}\xi^a\xi^b)^{1/2}_{|\gamma[I]}$. Pick

$$\epsilon = \inf_{\gamma[I]} \min \left\{ 1, \left(\frac{|h_{ab}\xi^a\mu^b|}{(h_{ab}\xi^a\xi^b)^{1/2}} - 1 \right)^2 \right\},$$

and consider any $g' \in B_0(g, \epsilon; h, C)$, where $\gamma[I] \subseteq C$. Writing $g'_{ab} = h_{am}\mu'^m h_{bn}\mu'^m - h_{ab}$, γ is g'-timelike just in case $|h_{ab}\xi^a\mu'^b|_{|\gamma[I]} > (h_{ab}\xi^a\xi^b)^{1/2}_{|\gamma[I]}$. Now, one can calculate that

$$h^{am}h^{bn}(g_{ab} - g'_{ab})(g_{mn} - g'_{mn}) = h_{am}h_{bn}(\mu^a\mu^b - \mu'^a\mu'^b)(\mu^m\mu^n - \mu'^m\mu'^n) = [h_{ab}(\mu^a - \mu'^a)(\mu^b - \mu'^b)]^2,$$

so putting $\eta^a = \mu'^a - \mu^a$ yields that $\sup_C h_{ab}\eta^a\eta^b < \epsilon$. The remaining calculations involve fields defined on $\gamma[I]$, so the subscript indicating as much will be omitted. It follows from this inequality and the Cauchy-Schwartz inequality that

$$|h_{ab}\xi^a\eta^b| \leq (h_{ab}\xi^a\xi^b)^{1/2}(h_{ab}\eta^a\eta^b)^{1/2} < (\epsilon h_{ab}\xi^a\xi^b)^{1/2} \leq (h_{ab}\xi^a\xi^b)^{1/2}, \tag{11}$$

where the last inequality uses the fact that, by definition, $\epsilon < 1$. Then the reverse triangle inequality entails that

$$|h_{ab}\xi^a\mu'^b| = |h_{ab}\xi^a(\mu^b + \eta^b)| \geq \big| |h_{ab}\xi^a\mu^b| - |h_{ab}\xi^a\eta^b| \big| = |h_{ab}\xi^a\mu^b| - |h_{ab}\xi^a\eta^b|,$$

where the last equality follows since $|h_{ab}\xi^a\mu^b| > (h_{ab}\xi^a\xi^b)^{1/2} > |h_{ab}\xi^a\eta^b|$ by the hypothesis and equation 11. Thus

$$|h_{ab}\xi^a\mu'^b| > |h_{ab}\xi^a\mu^b| - (\epsilon h_{ab}\xi^a\xi^b)^{1/2} \geq |h_{ab}\xi^a\mu^b| - (h_{ab}\xi^a\xi^b)^{1/2} \left| \frac{|h_{ab}\xi^a\mu^b|}{(h_{ab}\xi^a\xi^b)^{1/2}} - 1 \right| = (h_{ab}\xi^a\xi^b)^{1/2}.$$

Thus γ is g'-timelike, but g' was arbitrary so each element of $B_0(g, \epsilon; h, C)$ contains a closed timelike curve. Since $B_0(g, \epsilon; h, C)$ is open in every C^k compact-open topology, g must be stably chronology violating in each. $\square$

This result extends immediately to any topology finer that the compact-open topologies, but that extension is not needed to prove proposition 6:

Proposition 12. *For any manifold M, if $\dim(M) \geq 3$, then chronology violating spacetimes are generic in $L(M)$ for every compact-open topology.*

Proof. Every spacetime with a compact M contains timelike curves [Hawking and Ellis, 1973, Prop. 6.4.2, p. 189], so suppose M is non-compact. We construct an open dense subset of the non-chronological spacetimes as follows. First, select any neighborhood $N(g)$ of an arbitrary g, which must contain a set of the form $B_k(g, \epsilon; h, C)$. Letting $S = M - C$, by lemma 1 there is some $g' \in N(g)$ such that $g'_{|U}$ is isometric to a chronology violating region of Gödel spacetime for some compact $U \subset M - C$. By proposition 11, there is an open neighborhood of g' consisting only of chronology violating metrics. Let $A_k(g, N(g), B_k)$ be the union of all such open neighborhoods determined by the choices of g, $N(g)$, and $B_k(g, \epsilon; h, C)$, and consider $A = \bigcup_g \bigcup_{N(g)} \bigcup_{B_k} A_k(g, N(g), B_k)$. By construction, (1) every neighborhood $N(g)$ of each g contains an element of A, i.e., A is dense in $L(M)$; (2) A is open, being the union of open sets; and (3) A contains only chronology violating spacetimes. So by definition the chronology violating spacetimes are generic in $L(M)$. $\square$

Corollary 3. *For any manifold M, if $\dim(M) \geq 3$, chronological spacetimes are nowhere dense in $L(M)$ for each compact-open topology.*

Proof. The complement of the open dense subset A constructed above contains the chronological spacetimes and must be nowhere dense. Since subsets of nowhere dense sets are nowhere dense, the chronological spacetimes in particular are nowhere dense. $\square$

Both proposition 6 and its corollary then follow as special cases.

One may be struck by how few details of the closed timelike curves figure in the above proofs. Indeed, the crucial aspect of the compact-open topologies that allows this is the fact that its neighborhoods do not control the behavior of metrics beyond a compact set. Thus I suspect that one should be able to generalize the above propositions to characterize general classes of properties that are dense, generic, or nowhere dense in the compact-open topologies. However, the usual classification of spacetime properties into local and global [Manchak, 2011, p. 413] is too broad for these purposes. This classification takes a property P of a spacetime (M, g) to be local iff all spacetimes locally isometric to (M, g) also have P, and global otherwise. Thus both the topology of M and the existence of a closed causal curve are global properties, whereas only the latter has any hope of having an analog to proposition 6. The difference seems to be that the properties of concern here can be instantiated on co-compact subsets of M for proposition 4, concerning density, and on compact subsets for propositions 5 and 6, concerning stability and genericity.

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