Mass-Energy-Momentum in General Relativity. Only there because of Spacetime?

DRAFT VERSION.

Dennis Lehmkuhl, Oriel College, Oxford University.
Email: dennis.lehmkuhl@philosophy.ox.ac.uk
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Abstract

I describe how relativistic field theory generalises the defining property of material systems to possess mass to the requirement of them having a mass-energy-momentum density tensor $T_{\mu\nu}$ (energy tensor for short) associated with them. I argue that according to general relativity $T_{\mu\nu}$ is *not* an intrinsic property of matter, looking at how the energy tensor for a relativistic material system can be derived in a Lagrangian framework. It will become evident that the matter fields Φ alone are not sufficient for such a derivation. The metric field $g_{\mu\nu}$ plays a prominent role in obtaining the energy tensor of a material system, and occurs explicitly in a generic $T_{\mu\nu}$. Accordingly, since $g_{\mu\nu}$ represents the geometry of spacetime itself, the properties of mass, stress, energy and momentum should not be seen as intrinsic properties of matter, but as *relational* properties that material systems have only in virtue of their relation to spacetime structure.

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1 Introduction

In the Stanford Encyclopaedia for Philosophy, in the entry "Intrinsic and Extrinsic properties", Weatherson [2007] writes:

> I have some of my properties purely in virtue of the way I am. (My mass is an example.) I have other properties in virtue of the way I interact with the world. (My weight is an example.) The former are the intrinsic properties, the latter are the extrinsic properties.

The claim that mass is intrinsic is initially plausible. But concepts like 'mass' and 'weight' are surely not theory-independent. I will argue in this article that although the mass density ρ of a material object is arguably an intrinsic property in Newtonian physics, this is not the case for the property that takes over the role of ρ in general relativity, viz. the mass-stress-energy-momentum density tensor $T_{\mu\nu}$.[1] Nevertheless, the non-vanishing of an energy tensor $T_{\mu\nu}$ at a point will turn out to be a necessary and sufficient condition for this point to be occupied by matter.[2]

[1]For simplicity, I will often just speak of the 'energy-momentum tensor' or even just of the 'energy tensor' of a material system, rather than of a mass-stress-energy-momentum density tensor. Note that $T_{\mu\nu}$ is *not* a tensor density in the mathematical sense: like the scalar field ρ in Newtonian theory, it is a tensor that represents a *physical* density, rather a mathematical object that transforms as a tensor density.

[2]Friedman has already suggested that "In a relativistic context the proper representative of matter is not the mass-density ρ but the total stress-energy density $\mathbf{T}$"; see Friedman [1983], p.222. Friedman goes on to argue that the non-vanishing of $T_{\mu\nu}$ should be seen as the criterion for a spacetime point to be "occupied", while claiming that we should still speak of 'empty spacetime' if the (geo)metric tensor field $g_{\mu\nu}$ is the only non-vanishing tensor field defined on the spacetime manifold M. Hence, Friedman foreshadows the post-hole argument discussion, arguing that $(M, g_{\mu\nu})$ represents spacetime (cf. Hoefer [1996] and Pooley [forthcoming]), rather than *just* M (Earman and Norton [1987]).

Before I make this more precise, let us see how $T_{\mu\nu}$ enters the theory of general relativity (GR). It is right at the core of the theory, in the Einstein (field) equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa_E T_{\mu\nu} \quad , \tag{1}$$

which should be compared to the Poisson equation of Newtonian gravitational physics:

$$\nabla^2 \varphi = \kappa_N \rho \quad . \tag{2}$$

Within their respective theories, both the Poisson equation and the Einstein equations are supposed to describe how gravity, represented by the left side of the respective equation, and matter, represented by the right side, interact with each other.[3]

In the Poisson equation (2), we have the second derivative of the gravitational potential φ on the left-hand side and the mass density ρ on the right-hand side. For the Einstein equations (1), the left-hand side is formed by the Ricci curvature tensor $R_{\mu\nu}$, the Ricci scalar R and the metric tensor $g_{\mu\nu}$; the former two being defined in terms of the latter.[4]

I will argue that while in Newtonian theory ρ is a fundamental field representing an intrinsic property of matter, this is not so for $T_{\mu\nu}$ in GR: the energy tensor $T_{\mu\nu}$ is in important ways less fundamental than the metric field $g_{\mu\nu}$. Historically, an idea that the reverse was the case was very important

[3]The custom in relativity theory is to include radiation like the electromagnetic field when speaking of 'matter'; I will deliver a justification of this choice of words in section 2. Although this article is mostly about the right-hand sides of the above equations, one note on their left-hand sides: while the left-hand side of the Poisson equation describes the gravitational field just like any other field, the left-hand side of the Einstein equations is often claimed to describe *both* the geometry of spacetime and the gravitational field. In Lehmkuhl [2008], I discussed the ways in which this alleged double role can be understood, and so I will not rehearse the discussion here. I do indeed think that the main issues discussed do not depend on whether one sees the metric field $g_{\mu\nu}$ as representing the geometry of physical spacetime, as 'just another field, not intrinsically different from the electromagnetic field', or as both at once. I will sometimes call $g_{\mu\nu}$ 'the geometry of spacetime', but people who do not like that and the ontological flavour this carries should just substitute for it 'the gravitational field' or 'the metric field', without this altering the points made in this article.

[4]Both equations also contain coupling constants, $\kappa_N = -4\pi G$ and $\kappa_E = \frac{8\pi G}{c^4}$, where the latter is obtained by demanding that the Einstein equations should go over into the Poisson equation in the non-relativistic limit.

for Einstein up until 1921; the idea he called 'Mach's principle'. In Einstein [1918], p.38, he expresses this in the following way:[5]

> Mach's principle: The $[g_{\mu\nu}]$-field is *fully* determined by the masses of bodies. Since according to the results of the special theory of relativity mass and energy are the same, and since energy is formally described by the symmetric energy tensor $(T_{\mu\nu})$, Mach's principle says that the $[g_{\mu\nu}]$-field is constrained and determined by the energy tensor.

Einstein's formulation of Mach's principle is often taken as indicating his commitment to a Leibnizian/relationalist programme: spacetime was supposed to be secondary to material objects.[6] Famously, GR does not fulfil Mach's principle as defined above; for example, the original gravitational field equations (1) allow empty Minkowski spacetime as a solution, among many other matter-free solutions. Even the modified field equations in which Einstein introduced the cosmological constant λ in order for them to accord with Mach's principle turned out to allow for non-trivial solutions even if $T_{\mu\nu} = 0$.[7] Furthermore, the left-hand side of the Einstein equations represents only part of the geometric structure — the Ricci curvature $R_{\mu\nu}$ — whereas the Weyl curvature $C_{\mu\nu\sigma\omega}$ is only constrained but not determined by the energy tensor $T_{\mu\nu}$.

We will see that *even if* knowing the energy tensor did uniquely determine the geometric structure, and hence if Einstein's formulation of Mach's principle was fulfilled, there would still be no reason to regard matter as more fundamental than spacetime in the theory. In other words, the criterion of fundamentality embodied in Mach's principle will turn out not to be sufficient for matter to be more fundamental than spacetime. For the only thing

[5] *Machsches Prinzip*: Das G-Feld ist *restlos* durch die Massen der Körper bestimmt. Da Masse und Energie nach den Ergebnissen der speziellen Relativitätstheorie das Gleiche sind und die Energie formal durch den symmetrischen Energietensor $(T_{\mu\nu})$ beschrieben wird, so besagt dies, dass das G-Feld durch den Energietensor bedingt und bestimmt sei. (My translation.)

[6] In a footnote, Einstein points out that up to the publication of this article (in 1918), he had not distinguished between what he now called Mach's principle and the relativity principle. In Einstein [1918], p.38, the relativity principle is defined in the following way (which is quite different to his earlier formulations): "The laws of nature are statements about spatio-temporal coincidences; hence they find their only natural expression in generally covariant equations."

[7] See Hoefer [1994] for details.

that would really suggest that either matter or geometry was more fundamental would be if the existence of one was a requirement for the existence of the other but not vice versa. We will see that spacetime *can* exist without matter, whereas systems *cannot* possess mass-energy-momentum density without spacetime structure being in place.

I will start out in section 2 by arguing that what we *mean* (or should mean) by 'a material system' when taking into account relativistic field theory is a system for which the fields Φ describing its fundamental properties have a mass-energy-momentum tensor $T_{\mu\nu}$ associated with them and only thereby deserve to be called *matter* fields. In section 3, I will then show that an energy tensor $T_{\mu\nu}$ typically needs *both* a matter field Φ and the metric field $g_{\mu\nu}$ in order to be defined, and that hence *the* main property of matter depends on a relation between matter fields and spacetime structure; whereby mass-energy-momentum turns out to be a relational property of matter.[8]

2 The concept of matter in relativistic field theory

Back in the 17th century, there were two important opinions about the nature of matter, about the property that *made* something material. Descartes claimed that it was the property of being *extended*, whereas Newton argued that it was the property of possessing *mass*, or rather inertial mass. Physics followed the direction outlined by Newton. Arguably the reason for this development is that the physics Newton proposed and into which his idea about the nature mof matter was embedded simply was much more successful empirically than Descartes. But the idea also has a lot of intuitive force. Imagine standing in front of a house, an (unsurprisingly) spacially extended house. You stretch out your hand to get a hold of the doorknob — and your hand goes right through. You do not feel any resistance, no *inertia* and hence have no sense of touch. Would you think of it as a house? Maybe. But would you think of it as a *material* house? Probably not, you might think of it as a

[8]Section 4.2 will also discuss an alternative perspective, according to which mass-energy-momentum is an intrinsic property of systems *containing* matter. But here also, systems can only possess this property if they contain a metric field $g_{\mu\nu}$ and at least one matter field Φ, and if $g_{\mu\nu}$ and Φ stand in a certain relation that 'gives rise' to the system possessing mass-energy-momentum. The nature of the relation holding between matter fields and metric field is pinned down in section 3.

ghost house at best. Then again, imagine a particle, a point particle like the electron that has no (so far) measurable extension but which *does* possess mass. Do you think of it as material? Probably. For it can *push* things, it has mass and hence the potential to posses energy and momentum. (These colloquial statements will be made more precise in subsequent sections.)

Then, at the end of the 19th century, something surprising happened: one found that electromagnetic fields can *push* things as well! More precisely, Maxwell, Poynting, Heaviside and J.J. Thomson, building on earlier ideas by W.Thomson and Faraday, found that an electromagnetic field possesses energy and momentum (but no rest mass). After this, physicists came to think of electromagnetic fields as *something like* matter. Some even tried to *reduce* 'ordinary' matter to electromagnetic fields, thinking of the inertial mass possessed by an electron as something to be derived by assuming that it is only a particular configuration of an electromagnetic field. This was part of the electromagnetic programme of Abraham and Mie — ultimately unsuccessful but once one of the biggest hopes of theoretical physics.[9]

While the electromagnetic programme was still at its high-point, something surprising happened yet again. The special theory of relativity (SR) was discovered/created, and with it electromagnetic fields, particles and material continua (like extended bodies and fluids) could be described in a framework that made them look even more like the same kind of thing. For all the previously separate properties mentioned above could be encoded by the special-relativistic mass-energy-momentum density tensors.

2.1 A short and incomplete history of energy-momentum tensors

Einstein had formulated special relativity (Einstein [1905a]) and shown that an important consequence of it was the famous connection (or 'equivalence') between energy and inertial mass (Einstein [1905b]). Minkowski provided a 4-dimensional reformulation of the theory (Minkowski [1908a,b]), showing that the Lorentz-transformations are analogous to rotations in 4-dimensional spacetime. As a consequence, coordinates were written in a manner analogous to 4-vectors, with three space- and one time-component — and with

[9]For more on the positions of Newton and Descartes see Jammer [1961]; for a concise summary of Abraham's and Mie's programme and its relation to GR and unified field theories see Vizgin [1994].

them, all the fundamental quantities of the theory were rewritten in such a way. Most importantly for us, 3-momentum and mass/energy were shown to be components of a 4-vector with mass/energy occupying the 'time-slot' of the vector.

In pre-relativistic physics, it was already well-known that a momentum 3-vector and a scalar quantity describing mass and/or energy, while sufficient for describing point particles, were not sufficient for describing *continuum* matter. In order to do that, quantities describing energy densities, energy current densities, momentum densities and momentum current densities within the continuum were needed. In 1905, there was already a sophisticated non-relativistic theory describing these quantities for the paradigm cases of classical continuum matter — extended bodies with internal stress, fluids, and electromagnetic fields.

Conceptually, the natural first step to generalise the energy-momentum 4-vector of a free particle would have been to search for a quantity describing the energy-momentum properties of a system of many particles — i.e. to develop the *relativistic* dynamics of stressed bodies and fluids. However, in the early days, special relativity was very closely associated with electrodynamics, and so one of the first tasks Minkowski set himself was to give a 4-dimensional reformulation of relativistic electrodynamics (Minkowski [1908b]).

In doing so, he showed that the energy (current) density, momentum (current) density and stress associated with the electromagnetic field in pre-relativistic physics can be seen as components of a second-rank tensor $T_{\mu\nu}$, the mass-stress-energy-momentum tensor density of the electromagnetic field.[10] It was mostly through the work of Max von Laue that Minkowski's way of constructing an energy tensor was extended from electromagnetism to 'ordinary matter' like extended stressed bodies, incoherent dust, and relativistic fluids (von Laue [1911a,b]).[11] Just as Minkowski had done for electromagnetic fields, von Laue could show that the familiar quantities describing e.g. stress and energy of an extended body or a fluid could be regarded as components of a second-rank tensor $T_{\mu\nu}$ associated with the material system, while including the relativistic equivalence between mass and energy in the T_{00}

[10] Again, for simplicity, I will often just speak of the 'energy-momentum tensor' or even just of the 'energy tensor' of a material system, rather than of a mass-stress-energy-momentum tensor density.

[11] See also Norton [1992], especially p.28-33 for a summary of von Laue's development of a general energy-momentum tensor for stressed bodies and fluids.

component of this tensor.[12] I will call energy-momentum tensors obtained in such a manner *constructed* energy tensors.

The following paragraph from one of Einstein's unpublished manuscripts (written in 1912) makes evident the importance Einstein attributed to these results:[13]

> The general validity of the conservation laws and of the law of the inertia of energy [...] suggest that [the symmetric energy-momentum tensor $T^{\mu\nu}$ of a given system and the equation $f^{\mu} = \partial_{\nu} T^{\mu\nu}$ where f^{ν} is the external 4-force acting on the system] are to be ascribed a general significance, even though they were obtained in a very special case [i.e. electrodynamics]. We owe this generalisation, *which is the most important new advance in the theory of relativity*, to the investigations of Minkowski, Abraham, Planck, and Laue. [...] To every kind of material process we want to study, we have to assign a symmetric tensor $T_{\mu\nu}$.

Thus we see that it was established soon after special relativity was formulated that every kind of matter had an energy tensor associated with it, encoding the new relativistic concepts of mass, energy, momentum and stress. The difference between electromagnetic fields and 'ordinarily material' continua like fluids and extended bodies reduced to them having energy tensors of different form: for electromagnetic fields, the sum of the diagonal components (the trace) of $T_{\mu\nu}$ necessarily vanishes, for ordinary continua the trace of $T_{\mu\nu}$ is necessarily non-zero. Hence, it now seemed justified to really think of electromagnetic fields as a kind of matter.

When Einstein started to search for a relativistic theory of the gravitational field, it seemed natural to expect the newly found energy tensor to play the role of the 'source' of gravitational fields in much the same way as the mass density ρ_m played this role in Newtonian gravitation theory, and

[12]Lange [2001, 2002] and Flores [2005] have recently revived the discussion of what the 'equivalence' between mass m and energy E of a material system *really* means, both of them attacking the intuition that 'equivalence' should be understood as 'identity'. But both authors only investigate the question of what $E = mc^2$ means in the context of special relativistic particle mechanics, rather than looking at the full mass-energy-momentum tensors of more general material systems.

[13]See the Collected papers of Albert Einstein, Vol. 4, Document 1 (p. 63); my emphasis. Cf. also Janssen and Mecklenburg, p. 41-50, who elaborate how the transition "from the electromagnetic view of nature to relativistic continuum mechanics" was performed.

as the electric charge density ρ_e does in electrostatics. (In section 3, I will argue that $T_{\mu\nu}$ is not a source in quite the same way as ρ_m and ρ_e after all.)

Even though it is natural to expect $T_{\mu\nu}$ to play the role of the gravitational field source, it is not a priori so. Indeed, there once was a candidate for the correct relativistic theory of gravitation which did not have the full energy tensor on the right-hand side of the field equation, but only its trace T^μ_μ: Nordström's theory of gravitation, developed between 1912 and 1914.[14]

Hence in the context of searching for gravitational field equations, Einstein and Grossmann in their 1914 *Entwurf* theory put *constructed* energy tensors on the right-hand side of their candidate field equations, even after having *derived* the vacuum field equations from a Lagrangian approach (Einstein and Grossmann [1914], Einstein [1914]).[15] Hilbert, on the other hand, assumed that the total Lagrangian depends not only on the metric field $g_{\mu\nu}$ (corresponding, for both him and Einstein, to the gravitational potential), but also on the electromagnetic potential A_μ. Thus he postulated that this Lagrangian would ultimately describe both matter and gravitational fields. Hilbert was licensed to do this because he followed Mie's programme, which hypothesised that all matter could eventually be reduced to electromagnetic fields. Hilbert was criticised for this speculative assumption by Einstein [1916] and Weyl [1917], who rederived the energy tensor in a way similar to Hilbert, but without assuming that the fundamental matter fields were only electromagnetic in nature. Thus, they allowed matter fields to be of arbitrary rank and for them to possess symmetry properties different from the antisymmetric electromagnetic field tensor $F_{\mu\nu}$, and hence introduced the modern concept of a general matter field Φ.[16]

So whereas Einstein and Grossmann originally took their energy tensors 'off the shelf', Hilbert *derived* the energy tensor of the electromagnetic field from a Lagrangian that also gave the vacuum Einstein and the vacuum

[14] For a magnificent review article on Norström's theory and the exchange between Einstein and Nordström see Norton [1992].

[15] See especially p. 97-109 of Einstein and Grossmann [1914].

[16] Note that although Hilbert's assumption of the electromagncetic nature of matter is questionable as an assumption of principle, the restriction to the case where only electromagnetic fields are present in addition to gravitational ones became a field of intense study, and Hilbert's form of the field equations became known as the Einstein-Maxwell equations. Furthermore, some of Einstein's work on unified field theories in the 1920s can be read as involving Einstein making assumptions very similar to Hilbert's. Indeed, Hilbert saw Einstein's work on affine field theory as "a collosal detour via Levi-Civita, Weyl, Schouten, Eddington [...]" back to Hilbert's 1915 theory, cf. Sauer and Majer [2005].

Maxwell equations. Hilbert saw this result as "the triumph of axiomatics" (cf. Kichenassamy [1993], p. 196). The result seemed particularly remarkable because Hilbert, being a follower of Mie's programme, thought that only the energy tensor of the electromagnetic field was fundamental, i.e. he thought of von Laue's energy tensors describing fluids and extended bodies as merely effective, ultimately to be derived from the behaviour of electromagnetic fields.[17]

Even though this hope was not fulfilled, it was subsequently found that the energy tensors of all paradigmatic material systems formalisable in a Lagrangian framework can be derived in a similar manner: which seems remarkable enough. I will call energy tensors obtained (or even defined) in such a way *Hilbert energy tensors*,[18] and will review a modern version of the derivation scheme in section 3.[19]

2.2 Metaphysical matters

We started out by saying that we only think of something as material if we feel *resistance* when trying to touch it, and if we know that it could *push* other things given certain circumstances, i.e. if it has inertial mass and hence the potential to possess energy and momentum. Then we have found that all these concepts have been unified and generalised in relativity theory by the introduction of mass-energy-momentum density tensors $T_{\mu\nu}$, encoding these properties for particles, extended bodies, fluids, electromagnetic fields — ba-

[17]See Sauer [1999], especially p.555, and Brading and Ryckman [2008] for more details on this aspect of Hilbert's work, and Vizgin [1994] for a general summary of Mie's programme.

[18]This terminology pops up in certain texts when referring to the energy tensors derived with the help of Lagrangians, but it is not a universally used terminology.

[19]A more detailed account of energy-momentum tensors would now go on to describe the Noether theorems and currents, giving rise to the definition of so-called *canonical energy tensors* in special relativity. It would report in detail the discussion of whether canonical energy tensors and Hilbert energy tensors are after all equivalent, even though canonical energy tensors (in contrast to Hilbert tensors) are not necessarily symmetric but have to be symmetrized by a procedure first found by Belinfante and Rosenfeld [1940]. Very recently, Leclerc proposed a proof to show that canonical energy tensors and Hilbert tensors are indeed equivalent *if* matter fields do not couple to the derivatives of the metric field, while Saraví [2002] explicitly showed the equivalence between the canonical and the Hilbert energy tensor for the free electromagnetic field. Another important topic on the purely mathematical side are Petrov's and Serge's classification of second rank tensors, classifying both the Ricci and energy tensors in a rigourously algebraic manner; see Hall [2004].

sically everything we could think of as material. It seems that intuitively we *only* think of something as matter *if* there is mass, energy and/or momentum associated with it, and *if* we find these properties associated with something which we did not think of as matter beforehand, we will now do so (remember how we changed our opinion about the electromagnetic field). In other words: it seems we regard possession of mass-energy-momentum as an essential, rather than as an accidental, property of material systems.

The distinction can be introduced in the following way: An essential property of an object is a property that it *must* have, if it exists, while an accidental property of an object is one that it *happens to have* but that it could lack, even if it exists.[20]

Note that an essential property does not need to be a fundamental property, i.e. a property that cannot be defined in terms of other properties. Indeed, we will now see that despite the above, mass-energy-momentum density is *not* a fundamental property in relativistic field theory.

2.3 Energy tensors, matter fields and the metric Field

The fundamental properties of a relativistic field theory, properties that are not defined in terms of other properties, are represented by tensor fields like $F_{\mu\nu}$, ϕ and $g_{\mu\nu}$. We often speak of $F_{\mu\nu}$ as 'an electromagnetic field', of ϕ as 'a scalar field' and of $g_{\mu\nu}$ as 'a metric field'. Strictly speaking, this kind of talk is slightly misleading, for $F_{\mu\nu}$ does not represent the electromagnetic field 'as such', but a certain property (or set of properties) of the physical electromagnetic field: its amplitude, or electromagnetic *field strength*. However, electromagnetic field strength is *the* defining property of a physical electromagnetic field; something is an electromagnetic field if and only if it can be associated with a tensor field $F_{\mu\nu}$. Still, we should see $F_{\mu\nu}$ as describing fundamental properties of a *material system*, namely one that contains (possibly only) an electromagnetic field. But I will follow common custom and speak of $F_{\mu\nu}$ as 'a matter field', intending this name to be an abbreviation for saying that it describes fundamental properties of a material system. Note that the fundamental properties of a material system do not need to be describable by only one tensor field. The fundamental properties of a perfect fluid for

[20]See Robertson [2008]. Fine has criticised the definition of essentiality in terms of possible worlds in Fine [1994] and in Fine [1995], especially in the latter arguing that there are senses of essentiality that are not captured by this definition. I will will not enter this discussion here.

example are described by a triple of matter fields, (ρ, v^μ, p), where ρ is the proper density of the particles the fluid consists of, v_μ the velocity field describing the movement of every particle, and p the pressure field, giving the momentum an arbitrary fluid volume element 'feels' due to the movement of the rest of the fluid.

I have argued above that what *makes* us think of systems being described by $F_{\mu\nu}$ or (ρ, v^μ, p) as *material* systems is that they possess mass-energy-momentum (density). But whereas 'mass' was described as a fundamental property in Newtonian physics, this is not the case for mass-energy-momentum density in relativistic field theory: every $T_{\mu\nu}$ is defined *in terms of* the fundamental matter fields associated with the material system. But this is not enough: energy tensors also depend on the metric field $g_{\mu\nu}$!
For example, the energy tensor of a an electromagnetic field is

$$T_{\mu\nu} = \frac{1}{4\pi}(F_\mu{}^\lambda F_{\lambda\nu} + \frac{1}{4} g_{\mu\nu} F^{\sigma\lambda} F_{\sigma\lambda}) \quad , \tag{3}$$

while the energy tensor for a perfect fluid is given by

$$T_{\mu\nu} = (\rho + p) v_\mu v_\nu - p g_{\mu\nu} \quad . \tag{4}$$

Note that it is not possible in either case to 'absorb' $g_{\mu\nu}$ by raising or lowering indices of the other tensors involved. But even if it was, the metric tensor would still be there implicitly, not because we could always 'pull it out again', but because we can only have two tensors of the 'same name' (i.e. $F_\nu{}^\lambda$ and $F_{\sigma\lambda}$ rather than $F_\nu{}^\lambda$ and $H_{\sigma\lambda}$) with differing numbers of upper and lower indices if we have a unique isomorphism between the tangent space of the manifold and its dual. And we only have such an isomorphism if we have a metric tensor field $g_{\mu\nu}$ defined on the manifold.[21]

Although the latter point seems of a rather formal nature, the main point is not: mass-energy-momentum tensors do not just depend on the matter fields but also on the metric field. What does this mean?

As so often in field theory, the Lagrangian framework is a particularly convenient tool to search for an answer to this question. So I will now show how energy tensors can be derived in such a framework. It will become clear that in order for a material system to possess mass-energy-momentum, we need a certain *relation* between a matter field and the metric field: they need

[21] See Wald [1984], p. 22-25.

to couple in a certain way.[22]

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[22]Two things should be noted.

i) Not every system can be given (or at least: has been given so far) a Lagrangian formulation. This is particularly true for complex fluid systems. Hawking and Ellis [1973], p.69-71, give a Lagrangian description of an an isentropic perfect fluid, while DeFelice and Clarke [1990], p.195-198, even give one for a general perfect fluid. But more complex fluid systems, like the ones involving viscosity and heat transfer, have not to my knowledge been given a Lagrangian formulation, although their energy tensors too depend on the metric tensor. Either way, note that the Lagrangian framework is a means to discover the way energy tensors depend on the metric, just as it is a means to derive the equations of motion of a given system; neither the existence of equations of motion nor the dependence of an energy tensor on the metric depends on a Lagrangian derivation.

ii) On the other hand, there are particularly simple material systems which do not explicitly depend on a metric tensor. I know of two, 'normal dust' and 'null dust'. The former represents a collection of particles which do not collide with each other (and is hence identical with a pressureless perfect fluid), the latter represents a collection of not directly interacting light rays. The energy tensor of both systems has the form (cf. equation (4))

$$T^{\mu\nu} = \rho v^{\mu} v^{\nu} \quad . \tag{5}$$

In the case of 'normal dust', the velocity vectors have to be time-like, in the case of 'null dust' they have to be light-like. Although these energy tensors do not depend on the metric tensor explicitly, their definition nevertheless demands some spacetime structure to be in place. In the case of normal dust, one could even argue that we need full-blown metrical structure anyhow, because an essential part of the model is that the velocity vectors are normalised, $v_\mu v^\mu = -1$, and in order to normalise vectors we need a metric. (One may loosen the demands of the model so that only affine structure is needed, but no further loosening seems possible.) In the case of null dust, we do not seem to need metric or affine structure in order to obtain an adequate energy tensor for the system, but we do need conformal structure, for otherwise there would be no way of saying whether a vector is time-like, space-like or light-like. In GR, both affine and conformal structure are implied by metric structure, and even without GR they can always be extended to metric structure. In fact, it is not too surprising that we do not need full-blown metrical structure even for the simplest material systems — after all, even some gravitational phenomena can be described without recurrence to the metric. (I thank Robert Geroch, Erik Curiel, Stephen Lyle and John Norton for helpful and patient discussions of the issues summarised in this footnote.)

3 Deriving energy tensors

3.1 The basic ideas of Lagrangian field theory

The basic idea of Lagrangian field theory is to derive the equations of motion (also called the field equations) of a field Ψ by using the calculus of variations. One starts by assigning to the fields a *Lagrangian density* $\mathscr{L}$, or Lagrangian for short, with the help of which the *action* S can be defined:[23]

$$S = \int_{\Omega} \mathscr{L}(\Psi, \partial_\mu \Psi, \partial_\nu \partial_\mu \Psi, \dots) d\Omega \tag{6}$$

Ω is a compact region in spacetime, $d\Omega$ a 4-dimensional volume element. The field Ψ can be a tensor field of arbitrary rank (and potentially symmetric, anti-symmetric or asymmetric). The dots in the Lagrangian signify that in principle it could depend on derivatives of the fields of arbitrary order.

In general relativistic systems, there are two kinds of fields: the matter fields Φ and the metric/gravitational field $g_{\mu\nu}$. We will see that the matter fields can only represent material systems if there is a metric field as well — that it is only because of their relation to the metric field that they have a mass-energy-momentum tensor $T_{\mu\nu}$ associated with them.

It is almost always enough to allow the Lagrangian to depend only on the first derivatives of the matter fields Φ in order to derive their equations of motion — and because in GR the matter fields live in a curved spacetime and because we would prefer their equations of motion to be *manifestly* generally covariant, we will assume the Lagrangian to depend on the covariant derivatives $\nabla_\mu \Phi$ of the matter fields rather than on their partial derivatives $\partial_\mu \Phi$.[24]

This is not possible for the metric, however. In GR (in contrast to other gravitation theories) the metric $g_{\mu\nu}$ and the covariant derivative ∇_μ are not independent of each other, but the Christoffel symbol ${\Gamma^\nu}_{\mu\sigma}$, which gives the difference between a covariant derivative and a partial derivative, is defined in terms of first order partial derivatives of the metric. Indeed, this is a consequence of a postulate of GR (the postulate of 'compatibility of metric

[23] Some authors reserve 'Lagrangian' for the integral of the Lagrangian density over a spatial hypersurface. Note that strictly speaking $\mathscr{L}$ in equation (6) also depends on the coordinates x^μ, but it is common to omit writing out this dependence explicitly.

[24] See Hobson et al. [2006], p. 531, for details on this, and note that we have to treat Φ^a and $\nabla_\mu \Psi^a$ as independent variables.

and connection'), namely that $\nabla_\sigma g_{\mu\nu} = 0$, which ensures that the geodesics of a curved spacetime (given by the connection ${\Gamma^\nu}_{\mu\sigma}$) coincide with the geodesics of a local Lorentz geometry (given by the metric $g_{\mu\nu}$). So in the case of the metric field, we need the Lagrangian to depend on its partial derivatives — and indeed, in order to derive the Einstein field equations, we need it to depend on the first *and* second order derivatives of the metric. Hence, the action for a general relativistic system with one matter field Φ and the metric field $g_{\mu\nu}$ is[25]

$$S = \int_\Omega \mathscr{L}(\Phi, \nabla_\mu \Phi, g_{\mu\nu}, \partial_\sigma g_{\mu\nu}, \partial_\rho \partial_\sigma g_{\mu\nu}) d\Omega \tag{7}$$

In any case, the idea is that the Lagrangian is a scalar functional (up to a total divergence) of the dynamical fields present in the system under consideration, and that if one knows the Lagrangian, one can deduce the equations of motion for the fields in question. In order to do so, we define a *variation* of a matter field Φ by $\Phi \to \Phi + \delta\Phi$. (The variation for the metric field $g_{\mu\nu}$ is defined similarly below.) $\delta\Phi$ is also called the 'virtual displacement' of Φ, the idea being that the virtual displacement $\delta\Phi$ of a field configuration Φ gives us kinematically possible configurations infinitely close to Φ. (That these configurations are only *kinematically* possible also explains why they are called virtual: $\Phi + \delta\Phi$ will normally not be a solution to the dynamical equations of motion we aim to find for the field Φ.) A variation in Φ then 'induces' a variation in $\mathscr{L}$, and eventually in S (note that variation δ and integration $\int$ commute).[26] The first-order variation in S induced by the variation in Φ is then

[25] See Hobson et al. [2006], p.531, for this form of the Lagrangian and its justification. There are many textbooks on Lagrangian field theory; a good place to start is indeed Hobson et al. [2006], who deliver a very concise treatment in the context of GR. Furthermore, classics like Landau and Lifschitz [1989], Hawking and Ellis [1973], and Wald [1984] give excellent treatments of the role variational methods play in GR, whereas Lanczos [1986] delivers a wonderful systematic-historic discussion of variational principles in general. Furthermore, Poisson [2004] gives a detailed presentation of the boundary terms which have been found necessary in order to make the variational problem for the case of the Hilbert Lagrangian (17) well-defined.

[26] A more abstract way of defining the notion of a variation is found by considering a smooth one-parameter family of fields Φ_λ which start from Φ_0 and satisfy appropriate boundary conditions. A variation is then defined by the equation $\delta\Psi := \frac{d\Psi_\lambda}{d\lambda}|_{\lambda=0}$. For this way of defining variations see e.g. Wald [1984], p.450, and Hawking and Ellis [1973], p.65. For the definition of the variation used in the main text see for example Hobson et al. [2006], p.87 and 529. For the commutation relation between variation and integration

$$\delta S = \int_\Omega \left(\frac{\partial \mathscr{L}}{\partial \Phi} \delta\Phi + \frac{\partial \mathscr{L}}{\partial(\nabla_\mu \Phi)} \delta(\nabla_\mu \Phi) \right) d\Omega \quad . \tag{8}$$

We now impose Hamilton's principle, namely that the variation δS of the action be stationary under variations of the fields $\delta\Phi$. More precisely, we demand that $\delta S = 0$ for every variation $\delta\Phi$ inducing it, if the variation vanishes on the boundary $\partial\Omega$ of the volume element Ω. It can be shown that a necessary and sufficient condition for the action to be stationary is the fulfilment of the so-called *Euler-Lagrange equations* of the system, which are equivalent to the equations of motion of the fields given a suitable Lagrangian $\mathscr{L}$:[27]

$$\frac{\delta \mathscr{L}}{\delta \Phi} := \frac{\partial \mathscr{L}}{\partial \Phi} - \nabla_\mu \left(\frac{\partial \mathscr{L}}{\partial(\nabla_\mu \Phi)} \right) = 0 \tag{9}$$

where $\frac{\delta \mathscr{L}}{\delta \Phi}$ is called the *variational derivative*, or *Euler-Lagrange derivative*, of $\mathscr{L}$ with respect to Φ.

see Hobson et al. [2006], p.88, while commutation between variation and partial/covariant derivatives is proven on p.529 and p.531. Note also that Kichenassamy [1993] distinguishes three different kinds of variation, which he terms *functional variation*, *label variation* and *Lie variation*, of which he suggests that for example Hilbert [1915] and Einstein [1916] did not clearly distinguish between them. Label variation and Lie variation become essential in proving conservation laws; in this text, we will only need what Kichenassamy calls the functional variation.

[27] For a proof that the fulfilment of the Euler-Lagrange equations is a necessary and sufficient condition for the action to be stationary see for example Hawking and Ellis [1973], p. 65; and cf. Doughty [1990], who on p. 178 points out that this is only the case if the Euler-Lagrange equations are consistent. Note that it is not immediately clear why the Euler-Lagragnge equations *should* coincide with the equations of motion of a given material system. There is a way to make sense of it in the context of point-particle mechanics, where the Lagrange function L has a more immediate significance as the difference between kinetic and potential energy of a particle, which allows one to separate the Euler-Lagrange equations into a kinetic term and a potential term. Putting the former to the left-hand side and the latter to the right-hand side then coincides with Newton's second law in a natural way; cf. Lanczos [1986], p.118-119, and Feynman [1963], Volume II, Chapter 19 for more details on this. I am not aware of a similar 'naturalness-argument' in the case of field theories, where a Lagrange density $\mathscr{L}$ does not have a corresponding interpretation but becomes a more abstract mathematical object. Still, we know that as a matter of fact, the Euler-Lagrange equations for a field coincide with the equations of motions of the field, just like in particle mechanics; and one sensible viewpoint is surely to not see this as a miracle at all but as a result of our freedom to *choose* Lagrangians. However, compare footnote 30.

The Euler-Lagrange equations for the metric field can be obtained in a very similar manner. The only difference is that the Lagrangian depends on first *and second* order (partial) derivatives of the metric, while it depends only on first order (covariant) derivatives of the matter fields. Hence, in order to get the Euler-Lagrange equations for the metric field, we need to demand that i) $\delta S = 0$ for all variations $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$; ii) the variation of $g_{\mu\nu}$ *and* the first derivatives of the metric, $\partial_\sigma g_{\mu\nu}$, vanish on the boundary $\partial\Omega$ of the spacetime region Ω. We then get the following Euler-Lagrange equations:

$$\frac{\delta \mathscr{L}}{\delta g_{\mu\nu}} := \frac{\partial \mathscr{L}}{\partial g_{\mu\nu}} - \partial_\sigma \left(\frac{\partial \mathscr{L}}{\partial(\partial_\sigma g_{\mu\nu})} \right) + \partial_\rho \partial_\sigma \left(\frac{\partial \mathscr{L}}{\partial(\partial_\rho \partial_\sigma g_{\mu\nu})} \right) = 0 \quad , \tag{10}$$

where again $\frac{\delta \mathscr{L}}{\delta g_{\mu\nu}}$ is called the *variational derivative* of $\mathscr{L}$ with respect to $g_{\mu\nu}$. We see that for every field to be varied we get a separate set of Euler-Lagrange equations.

3.2 Equations of motion / field equations

Before we look at how we get the energy tensor $T_{\mu\nu}$ of a material system out of this procedure, let us look at *particular* field equations obtained by the above schema. If the only matter field present in the system is a free scalar field ϕ, then we find that the Lagrangian

$$\mathscr{L}_\phi = \frac{\sqrt{-g}}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \tag{11}$$

(with $g := det(g_{\mu\nu})$) gives us the following field equations when we set $\mathscr{L} = \mathscr{L}_\phi$ and $\Phi = \phi$ in equation (9):

$$\nabla^\mu \phi \nabla_\nu \phi = 0 \quad . \tag{12}$$

If, on the other hand, the only matter field present is a free electromagnetic field $F_{\mu\nu}$ defined in terms of a vector potential A_μ by $F_{\mu\nu} = \nabla_\nu A_\mu - \nabla_\mu A_\nu = \partial_\nu A_\mu - \partial_\mu A_\nu$, we find that taking

$$\mathscr{L}_{EM} = \frac{\sqrt{-g}}{8\pi} g^{\alpha\gamma} g^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta} \quad , \tag{13}$$

gives us the source-free Maxwell equations

$$\nabla_\mu F^{\mu\nu} = 0 \tag{14}$$

when put into equation (9) and varying $\mathscr{L}$ with respect to A_μ. The Maxwell-equations with a 4-current density j_μ as a source can be derived similarly.[28]

While the Maxwell equations describe the dynamics of the free electromagnetic field $F_{\mu\nu}$, the vacuum Einstein equations

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{16}$$

describe the dynamics of the free metric field $g_{\mu\nu}$. We obtain them by putting the so-called Hilbert-Lagrangian into equation (10), varying $\mathscr{L}_G$ with respect to the metric $g_{\mu\nu}$:[29]

$$\mathscr{L}_G = \sqrt{-g} R \quad , \tag{17}$$

where R is the Ricci curvature scalar. Remember that the latter is the twice contracted Riemann curvature tensor, and hence defined purely in terms of the metric tensor and its first two derivatives.[30]

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[28]Note that only this half of the free Maxwell equations (corresponding to the three-vector equations $\vec{\nabla} \cdot \vec{E} = 0$ and $\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 0$) is a dynamical equation, the second half

$$\nabla_\sigma F_{\mu\nu} + \nabla_\mu F_{\nu\sigma} + \nabla_\nu F_{\sigma\mu} = 0 \tag{15}$$

follows from the definition of $F_{\mu\nu}$ in terms of the vector potential, and corresponds to the three-vector equations $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$.

[29]Note that if we vary the metric field $g_{\mu\nu}$, the volume element Ω that occurs in equation (7) is affected by the variation due to its dependence on the metric. Wald [1984], p.452-3, describes how to handle this. The main practical consequence is that the term $\sqrt{-g} = \sqrt{-det(g_{\mu\nu})}$ needs to occur in every Lagrangian dependent on $g_{\mu\nu}$, so that $\mathscr{L}$ becomes a scalar density and thereby does not depend on the volume element.

[30] Arguably, the whole procedure of obtaining equations of motion in this way looks a bit like black magic and trickery — it seems we freely *choose* the Lagrangians in such a way that they give us the equations of motions we already know. But actually, due to symmetry principles there is much less arbitrariness involved than appears at first sight: Lovelock [1969] proved that in 4 dimensions any Lagrangian that i) gives rise rise to field equations of second order in the metric; and ii) is invariant under arbitrary coordinate transformations *must* be equivalent to the Hilbert Lagrangian (plus an extra term if the cosmological constant is non-zero). Lovelock [1974] proves a similar result for the Maxwell equations, while Grigore [1992] generalises Lovelock's original result to arbitrary dimensions of spacetime. However, it does not seem possible to find such uniqueness results for any system of fields: there are examples where different Lagrangians give the same equation of motion; cf. Trautmann [1962].

3.3 Enter the energy tensor

In the above, I have followed common custom and spoken of equations (11), (13) and (17) as the field equations of 'free' scalar, electromagnetic and metric fields, respectively. Indeed, the Lagrangian for the metric field depends only on the metric field and its derivatives. But the Lagrangians for the *matter* fields ϕ and $F_{\mu\nu}$ depend not only on these fields and their derivatives but *also* on the metric field $g_{\mu\nu}$!

Somewhat surprisingly, it is only via the dependence of matter Lagrangians on the metric field $g_{\mu\nu}$ that we can derive what *makes* us think of them as matter fields: an assoicated mass-energy-momentum tensor $T_{\mu\nu}$. For example, the variational derivative of the electromagnetic Lagrangian with respect to the metric field, $\frac{\delta \mathscr{L}_{EM}}{\delta g_{\mu\nu}}$, gives us a generalised version of Minkowski's energy tensor of the electromagnetic field:[31]

$$T_{\mu\nu} = \frac{1}{4\pi}(F_{\mu}{}^{\lambda} F_{\lambda\nu} + \frac{1}{4} g_{\mu\nu} F^{\sigma\lambda} F_{\sigma\lambda}) \quad . \tag{18}$$

So it is *not just* the gravitational Lagrangian $\mathscr{L}_G$ which depends on the metric field $g_{\mu\nu}$, but also the electromagnetic Lagrangian $\mathscr{L}_{EM}$, giving us a total Lagrangian of the form

$$\mathscr{L}_{\text{total}} = \mathscr{L}_G(g_{\mu\nu}, g_{\mu\nu,\sigma}, g_{\mu\nu,\sigma\omega}) + \mathscr{L}_{EM}(g_{\mu\nu}, F_{\mu\nu}) \quad . \tag{19}$$

Indeed, it is this Lagrangian that gives us the full *gravitational* field equations if we put (19) into the Euler-Lagrange equations for the metric field $g_{\mu\nu}$ (10). They then give us:

$$\left(G_{\mu\nu} = \frac{\delta \mathscr{L}_G}{\delta g^{\mu\nu}}\right) = \left(-\frac{\delta \mathscr{L}_{EM}}{\delta g^{\mu\nu}} = T_{\mu\nu}\right) \quad . \tag{20}$$

These equations are, up to the coupling constant, equivalent to the Einstein field equations (1) with electromagnetic fields being the only kind of matter present.[32]

[31] It is a generalised version because Minkowski formulated his energy tensor for special relativistic systems, whereas this energy tensor, allowing for a dynamical metric, holds in general relativity as well.

[32] See Wald [1984], p.455-6, about how the coupling constant can be included into the definition of $T_{\mu\nu}$ presented below; many other authors keep the constant independent of the definition of $T_{\mu\nu}$.

The first thing to note is that a similar procedure is available for every matter field Φ: given a Lagrangian that gives us the equations of motion for Φ, the same Lagrangian will give us the energy tensor of the field Φ, when the variational derivative with respect to the metric field $g_{\mu\nu}$ is taken. This procedure turns out to be so successful in recovering energy tensors previously constructed by phenomenological means that in mathematical physics

$$T_{\mu\nu} := \frac{2}{\sqrt{-g}} \frac{\delta \mathscr{L}(g_{\mu\nu}, \Phi, \nabla_\mu \Phi)}{\delta g^{\mu\nu}} \tag{21}$$

is used as the *definition* of the energy tensor of a newly introduced matter field Φ.[33]

Note that it follows from equation (21) that the energy tensor of a material system vanishes if *and only if* the corresponding matter field Φ vanishes. For the latter tells us that $T_{\mu\nu}$ can only vanish in a certain region if Φ vanishes in the same region (for $g_{\mu\nu}$ never vanishes), and it tells us that if $T_{\mu\nu}$ vanishes then Φ must vanish also. As a consequence, every non-vanishing matter field Φ brings with it a non-vanishing energy tensor $T_{\mu\nu}$ and vice versa.[34] Hence, the presence of a non-vanishing energy tensor $T_{\mu\nu}$ at a point is a necessary and sufficient condition for this point to be occupied by matter.[35]

So the variational derivative of the Lagrangian of a matter field Φ with respect to the metric field $g_{\mu\nu}$ is the energy tensor of the field Φ. We thus see that the presence of the metric in matter Lagrangians is truly essential: without metric dependence of the Lagrangian, $\frac{\delta \mathscr{L}_\Phi}{\delta g^{\mu\nu}}$ would be trivially zero.[36]

Equation (20) and (21) also show us why the energy tensor $T_{\mu\nu}$ is not a 'source' of the metric field in the same way as the mass density ρ_m is a source

[33] Misner et al. [1973] (and many other books) define the energy tensor in this manner, Wald [1984] (and equally many books) define $T_{\mu\nu}$ as the variational derivative of the action S, rather than the Lagrangian $\mathscr{L}$, with respect to the metric $g_{\mu\nu}$. But given that integration and variation commute, the two definitions are equivalent.

[34] Cf. Hawking and Ellis [1973], p.61.

[35] 'Non-vanishing' means that some of the components of the energy tensor are non-zero in a given coordinate system at a point. And if some of the components of a tensor are non-zero in one coordinate system, then this implies that there are non-zero components in *every* coordinate system.

[36] Brown and Brading [2002] put forward a complementary argument, suggesting that general covariance (and the role it plays in conservation laws as obtained from Noether's theorems) rules out the possibility of having a Lagrangian that only depends on the matter fields but not on the metric field. See in particular page 3 and section V of the above cited paper.

of the gravitational field in Newtonian theory, or as the electric charge density ρ_e is a source in electrostatics. In both cases, one can specify the source first and then obtain the gravitational or the electric field, respectively, via the field equations. This is not possible for the energy tensor $T_{\mu\nu}$, which already depends on the field $g_{\mu\nu}$ for which it is supposed to act as a source.[37]

But why don't we interpret the Einstein tensor $G_{\mu\nu} = \frac{\delta \mathscr{L}_G}{\delta g^{\mu\nu}}$ as the energy tensor of the metric/gravitational field? Indeed, such a proposal was made soon after the birth of GR. Lorentz [1916] argued that rather than following Einstein, who introduced a pseudo-tensor as the representative of the energy-momentum of the gravitational field, we should regard the Einstein tensor $G_{\mu\nu}$ as representing the energy tensor of the gravitational field. The Einstein equations (1) would then primarily describe an interchange of mass-energy-momentum between the gravitational field on the one hand and the matter fields on the other hand. But even if one agrees with Lorentz that an energy-tensor for the metric field is very desirable, his way of looking at things is arguably unattractive due to the different nature of the conservation laws for $T_{\mu\nu}$ on the one hand and for $G_{\mu\nu}$ on the other. In short, $\nabla^\mu G_{\mu\nu} = 0$ is an algebraic identity, whereas $\nabla^\mu T_{\mu\nu} = 0$ is a weak conservation law, in the sense that it holding depends on the field equations holding. But I will not develop this argument here.[38]

For the topic at hand the crucial lesson is that it is only via the dependence of the matter Lagrangian on *both* the matter field Φ *and* the metric field $g_{\mu\nu}$ that we can define an energy tensor for the material system represented by Φ.[39]

I will now show in more detail that the energy tensor $T_{\mu\nu}$ of matter is definable because of the way the matter fields and the metric field *relate* to each other.

[37]Wald [1984], p. 73, makes the same point. It should be noted that the blurring of the distinction within GR between a field and its source is a point quite separate from the non-linearity of the field equations; which is the failure of a superposition principle holding for the solutions of the field equations, and a feature the Einstein equation possess even in the vacuum case.

[38]See Trautmann [1962] for details.

[39]The energy tensor $T_{\mu\nu}$ *would* be definable if the total matter Lagrangian $\mathscr{L}_M$ could be decomposed into two terms: $\mathscr{L}_M = \mathscr{L}_{M_1}(g_{\mu\nu}, \Xi) + \mathscr{L}_{M_2}(\Phi)$, where Φ and Ξ are supposed to be matter fields. But it would then be the energy tensor of only the field Ξ, and hence Φ would not be a matter field after all.

3.4 Mixed Lagrangians

Imagine that there are only two non-gravitational fields in the world, say a scalar field ϕ and an electromagnetic field $F_{\mu\nu}$, with Lagrangians

$$\mathscr{L}_\phi = \frac{\sqrt{-g}}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \tag{22}$$

and

$$\mathscr{L}_{EM} = \frac{\sqrt{-g}}{8\pi} g^{\alpha\gamma} g^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta} \tag{23}$$

respectively. Then there is a clear distinction between the case where the two fields are interacting directly and the case where they are not interacting directly. What is more, both cases are physically possible! For the total Lagrangian of the physical system could either be

$$\mathscr{L}_1 = \mathscr{L}_\phi + \mathscr{L}_{EM} \tag{24}$$

or

$$\mathscr{L}_2 = \mathscr{L}_\phi + \mathscr{L}_{EM} + \mathscr{L}_{\text{int}} \tag{25}$$

where $\mathscr{L}_{\text{int}}$ describes the direct interaction between the two fields. For example, it could be

$$\mathscr{L}_{\text{int}} = \sqrt{-g}\omega^2 \phi F_{\mu\nu} F^{\mu\nu} \tag{26}$$

where ω is the coupling constant. In this case, ϕ and $F_{\mu\nu}$ directly and minimally couple to each other.

Two fields are said to *directly couple* if they are multiplied with each other in the Lagrangian, giving coupled differential equations as field equations via the Euler-Lagrange equations. They are said to *indirectly couple* if they do not couple directly but via an intermediate field: if field ϕ directly couples to field $g_{\mu\nu}$ and field $g_{\mu\nu}$ directly couples to field $F_{\mu\nu}$, then field ϕ indirectly couples to field $F_{\mu\nu}$. Another example is equation (24): the scalar field ϕ and the electromagnetic field $F_{\mu\nu}$ indirectly couple to each other via the metric field $g_{\mu\nu}$.[40]

[40]Yet another example of indirect coupling is Brans-Dicke theory in the Jordan frame (cf. Weinstein [1996]): the newly introduced scalar field ϕ indirectly couples to all matter fields via directly coupling to the metric $g_{\mu\nu}$; which in turn directly couples to all matter fields.

A field ϕ is said to *minimally couple* to a field $F_{\mu\nu}$ if it couples only to $F_{\mu\nu}$ itself, rather than to a tensor formed of its derivatives.[41]

Now imagine a world in which we only have an electromagnetic field $F_{\mu\nu}$ and a metric field $g_{\mu\nu}$. In this case, it is *not* possible to write the Lagrangian in the form

$$\mathscr{L}_{tot} = \mathscr{L}_G + \mathscr{L}_{EM} + \mathscr{L}_{\text{int}} \quad , \tag{27}$$

where only $\mathscr{L}_{\text{int}}$ would contain both fields. For as we have seen in the last subsection, $\mathscr{L}_{EM}$ does *already* contain the metric field $g_{\mu\nu}$, *and* in order for the electromagnetic field to have mass-energy-momentum, $\mathscr{L}_{EM}$ *must* indeed depend on $g_{\mu\nu}$. And so $\mathscr{L}_{EM}$ exactly fulfils our definition of what it means for one field to directly and minimally couple to another field: $g_{\mu\nu}$ and $F_{\mu\nu}$ so couple in $\mathscr{L}_{EM}$. Hence, what would normally be the interaction term between metric and electromagnetic field, $\mathscr{L}_{\text{int}}(g_{\mu\nu}, F_{\mu\nu})$, is already and necessarily part of the 'free' electromagnetic Lagrangian $\mathscr{L}_{EM}$. Thus, the way the metric field and the electromagnetic field are related to each other results in the electromagnetic field having an energy tensor! A similar argument can be made for every matter field that is described in the Lagrangian framework.[42]

The above should not be misunderstood as saying that energy tensors result from an *interaction* between the matter fields Φ and the metric field $g_{\mu\nu}$. For an interaction demands that all fields present are dynamical fields, and so such a statement would cause trouble with the fact that we can define energy tensors in special relativity (with its non-dynamical metric) equally well as in general relativity (with its dynamical metric). And even in special relativity can we obtain the energy tensor as the variational derivative of the matter Lagrangian, $T_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta \mathscr{L}_\Phi(g_{\mu\nu}, \Phi, \Phi_{;\mu})}{\delta g^{\mu\nu}}$; for the definition of the

[41]This definition of minimal coupling is a generalisation of the principle of minimal coupling proposed by Goenner [1984], which is about minimal coupling to the metric field in particular. (He argues that the common definition of minimal coupling, namely that general relativistic field equations can be obtained from special relativistic ones by demanding that $\eta_{\mu\nu} \to g_{\mu\nu}$ and $\partial_\mu \to \nabla_\mu$, where $\eta_{\mu\nu}$ is the Minkowski metric, is unique only for *first*-order differential equations.) In the case of a matter field $F_{\mu\nu}$ coupling to the metric field $g_{\mu\nu}$, the relevant tensor to which $F_{\mu\nu}$ would have to couple for there to be non-minimal coupling is the Riemann tensor $R_{\mu\nu\sigma}{}^{\omega}$. Interestingly, this definition of minimal coupling also means that if we regarded the electromagnetic vector potential A_μ, rather than the Faraday tensor $F_{\mu\nu}$, as representing the electromagnetic field on a fundamental level, equation (26) would have to be seen as expressing non-minimal coupling between the scalar field ϕ and the electromagnetic field A_μ.

[42]The direct coupling to the metric also occurs in both $\mathscr{L}_1$ and $\mathscr{L}_2$ above.

variation $\delta\mathscr{L}$ as a result of a 'virtual displacement' of $g_{\mu\nu}$ (see the beginning of this section) does *not* depend on the metric field being dynamical or non-dynamical.[43] Of course, we are not allowed to apply Hamilton's principle to the metric field of special relativity, for otherwise it would be a dynamical field. This is a fortunate prohibition, for if we were allowed to demand $\delta S = 0$ for all $\delta g_{\mu\nu}$ in special relativity, it would give us $T_{\mu\nu} = 0$ as a consequence, precisely because there is no purely metric Lagrangian in SR. And of course we know that we can (and normally do) have non-vanishing energy tensors even in special relativistic systems. So even though the matter fields Φ and the metric field $g_{\mu\nu}$ *are* interacting in GR, they do not need to be interacting in order for there to be a non-vanishing energy tensor. What we *do* need in order to have a non-vanishing energy tensor for every matter field a weaker requirement: *coupling* of every matter field to the metric field in the matter Lagrangian. Indeed, it seems sensible to make a distinction between speaking of two fields *interacting* and two fields *coupling*. For a dynamical field can couple to a non-dynamical field in the way defined above, but we would not speak of an *inter*action if only one of the two fields was dynamical: a non-dynamical field acts without being acted upon if it couples to a dynamical field. Hence, two fields interacting should be seen as sufficient but not necessary for the fields to couple, whereas two fields coupling is necessary but not sufficient for two fields to interact.[44]

[43]See Landau and Lifschitz [1989], p. 293, for a similar statement.

[44]In the 'constructive' approach towards special relativity, it has been argued that the metric should not be seen as an entity in its own right, but as merely encoding certain properties of the matter fields; see Brown [2005] and Brown and Pooley [2004] for an advocacy and Norton [2007] and Janssen [2007] for a criticism of this position. One reason for advocating the position is that it is seen as an unsatisfactory aspect of special relativity that the metric seems to violate the action-reaction principle by acting without being acted upon; a feature we get rid of if the metric is conceived as encoding properties of the fully interacting matter fields. What I say above does not help to decide for or against a constructive approach, although constructivists and anti-constructivists will see it in very different lights. The constructivist will see the dependence of energy tensors on the metric tensor as yet another example of the fundamental difference between special and general relativity: he will say that in SR we have energy tensors depending on something that can be reduced to the properties of the matter fields and that hence does not have a relational character, whereas with respect to GR he will say that the energy tensor of a given matter field depends on its relation to a new dynamical field, the metric field. The anti-constructivist, on the other hand, will stick with his more unified picture and say that in both GR and SR energy tensors depend on the structure of spacetime, be it dynamical or not.

To sum up, the matter fields Φ stand in a certain *relation* to the metric field $g_{\mu\nu}$, they *couple directly* to $g_{\mu\nu}$, a relation necessary for there to be mass-energy-momentum $T_{\mu\nu}$.

4 What kind of property is mass-energy-momentum density?

In section 1, I pointed out that the only thing that would really speak to either matter or geometry being more fundamental would be if one was a requirement for the existence of the other but not vice versa.[45] Both the (geo)metric field $g_{\mu\nu}$ and the matter fields Φ are fundamental fields according to GR: neither requires other fields, and in particular not the respective other, in order to exist / to be defined. But the tensor field that is supposed to represent *the* main properties of matter, the mass-energy-momentum density tensor $T_{\mu\nu}$, is a non-fundamental field, and it *does* require *both* $g_{\mu\nu}$ and Φ by being defined in terms of them.

So can we really say that $T_{\mu\nu}$ is an intrinsic property of material systems, as was the mass density ρ in Newtonian mechanics? Should it not be a necessary condition for $T_{\mu\nu}$ to depend only on the matter fields Φ, rather than on both Φ and $g_{\mu\nu}$, in order for it to be an intrinsic property of matter?

4.1 A relational property?

Lewis [1983], p.111-2, introduces the distinction between intrinsic and extrinsic properties in the following way:

> A sentence or statement or proposition that ascribes intrinsic properties to something is entirely about that thing; whereas an ascription of extrinsic properties to something is not entirely about that thing, though it may well be about some larger whole which includes that thing as part. A thing has its intrinsic properties in virtue of the way that thing itself, and nothing else, is.

[45] Note that A requiring B while B not requiring A is *sufficient*, but not necessary, for B to be more fundamental than A. For if A did not require B but C, while B required neither A nor C, then B would also be more fundamental than A. And if A, B, C are fields, then according to the terminology introduced in the preceding paragraph, B would be a fundamental field and A a non-fundamental field.

> Not so for extrinsic properties, though a thing may well have these in virtue of the way some larger whole is. The intrinsic properties of something depend only on that thing; whereas the extrinsic properties of something may depend, wholly or partly, on something else.

We have seen in section 3 that every energy tensor $T_{\mu\nu}$ depends on both the matter fields Φ and the metric field $g_{\mu\nu}$, and that it is only via the coupling between metric field and matter fields that we can define a mass-energy-momentum tensor $T_{\mu\nu}$ for a given material system.

It seems to follow that the energy tensor $T_{\mu\nu}$ must be seen as corresponding to an *extrinsic* property of material systems. For matter only has an energy tensor $T_{\mu\nu}$ associated with it *in virtue* of the relation between the matter fields Φ and the metric field $g_{\mu\nu}$.

We should now distinguish between relations and relational properties, in order to find out what kind of extrinsic property *exactly* we should see $T_{\mu\nu}$ to be, and to see what it is a property *of*.

An n-place relation $G(x_1, \ldots, a_n)$ depends on n entries. An example of a 2-place relation is '$Rxy :=$ (x is the father of y)'. A relation gives rise to a set of relational properties. For example, the fact that Hermann Einstein and Albert Einstein stand in a certain relation to each other (namely a father-son relation) gives rise to the relational property of Hermann that he is a father, and to Albert's relational property of being a son.

We can make this distinction more precise in the following way: let us call an n-place predicate, where $n \geq 2$, a *relation* if and only if the predicate contains only variables x, y, z, but no designators a, b, c. Let us speak of a *relational property* if we have an n-place predicate that contains either a mixture of variables and designators or only designators.

Then we can think of Hermann and Albert Einstein in the following way. We can think of them as two individuals (or two systems each of which consists of only one individual) who stand in certain relations to each other and have hence various relational properties. For example, if we denote the (antisymmetric) relation 'x is a son of y' as Sxy, Albert as a and Hermann as h, then the fact that Albert possesses the relational property of 'being the son of *someone*' can be expressed by the sentence $\exists y Say$, while the fact that Albert possesses the relational property of 'being the son of Hermann Einstein' can be expressed by the sentence Sah. Both properties are properties of Albert *only*, even though they are due to his standing in a certain relation to Hermann.[46]

Rather than looking at the system consisting only of Albert Einstein and wondering about the relational properties he has in virtue of standing in a certain relation to his father, we could also look at the bigger system (Albert, Hermann) $= (a, h) := s$ and wonder about the property the system s has in virtue of the relations its parts stand in. One *intrinsic* property s has (a property that only depends on s itself, but not on the relation s has to anything else) is that its elements stand in a father-son relationship. Let us denote this $1-$place property of systems by Fx, and the fact that s possesses this intrinsic property as being expressed by the sentence Fs.

But not much seems to depend on whether we choose the first or the second perspective. In both cases we can say everything we want to say about Hermann and Albert Einstein in this context, and in both cases the important fact is that the relation between the two leads to certain properties. Still, it is an interesting question whether the energy tensor $T_{\mu\nu}$ is more like Albert's relational property of being Hermann's son, Sah (case 1), or whether it is more alike the property of the system (Albert, Hermann), that its parts stand in a father-son relationship, Fs (case 2).

In the second case, we regard the predicate $\mathscr{T}_1(x)$, where the domain of x is 'systems in which both a metric field $g_{\mu\nu}$ and at least one matter field Φ is defined', as expressing the intrinsic property of such systems to have an energy tensor associated with them, a property they have because of the relation that holds between their two kinds of parts. The system $s := (g_{\mu\nu}, \Phi)$ possesses the intrinsic property $\mathscr{T}_1 s := T_{\mu\nu}(g_{\mu\nu}, \Phi)$ in virtue of the metric field $g_{\mu\nu}$ and the matter fields Φ standing in a certain relation $\mathscr{T}_1(x)$.

In the first case, we would say that mass-energy-momentum density $T_{\mu\nu}$ is a property of only the material parts of the total system, represented by the matter fields Φ, a property that the material system has in virtue of its relation to the metric field $g_{\mu\nu}$. We would then say that although $T_{\mu\nu}$ *depends* on the relation the matter field Φ has to the metric field $g_{\mu\nu}$ (like Albert's property of being the son of Hermann depends on his relation to Hermann), the property 'possessing mass-energy-momentum' is still a property *only of* matter, just as it is a (relational) property of only Albert that he is the son of Hermann. We can reformulate this by calling $\mathscr{T}_2(x, y)$, where the domain

[46] I have in this paragraph made a distinction between a sentence Sah and the property of a it describes. For ease of reading, I will not make this distinction explicit from now on and just call Sah itself a property of a.

of x is 'metric fields' and the domain of y is 'matter fields', the relation which allows us to express the fact that a material system represented by Φ has the relational property of having an energy tensor associated with it as $\mathscr{T}_2 g_{\mu\nu}\Phi := T_{\mu\nu}(g_{\mu\nu}, \Phi)$. The trouble with this is only that nothing in the representation $\mathscr{T}_2 g_{\mu\nu}\Phi$ gives away that $\mathscr{T}_2$ is a property of the part of the total system represented by Φ only — but that is also the case for using Sah in order to represent system Albert's property of being system Hermann's son.

As in the case of Hermann and Albert, I do not think that much depends on the perspective we choose. Indeed, one could arguably represent both $\mathscr{T}_1$ and $\mathscr{T}_2$ as a map from the set of pairs $(g_{\mu\nu}, \Phi)$ to the set of all energy-momentum tensors, $\mathscr{T} : (g_{\mu\nu}, \Phi) \longrightarrow T_{\mu\nu}(g_{\mu\nu}, \Phi)$.[47] The distinction would merely lie in seeing the set $(g_{\mu\nu}, \Phi)$ as an 'object' or as a pair of objects. Either way, we find that material systems possessing (non-vanishing) mass-energy-momentum tensors depends on a *relation* between the matter fields and the metric field.[48]

4.2 Relational *and* essential?

In section 2.2 I have argued that we treat possession of mass-energy-momentum as an essential property of material systems. Now we have seen in the previous section that it is also a *relational* property, a property material systems only have in virtue of their relation to spacetime structure.

It might be claimed that every essential property must be an intrinsic property. Indeed, the claim can be argued to follow from Lewis' principles of recombination and plenitude for the creation of possible worlds, if recom-

[47]Of course, this is only the simplest case: if a system contains m matter fields, then $\mathscr{T}$ is an $m+1$-relation: $\mathscr{T} : (g_{\mu\nu}, \Phi_1, \ldots, \Phi_m) \longrightarrow T_{\mu\nu}(g_{\mu\nu}, \Phi_1, \ldots, \Phi_m)$.

[48]Note that there is still the age-old question of whether the metric/gravitational field could possess energy-momentum in its own right, and in the same way as the matter fields. Given what I said above, the success of such a project would call into question the very distinction between 'matter fields' like Φ on the one hand and 'geometric fields' like $g_{\mu\nu}$ on the other hand, and hence be an important step towards a unified field theory. But alas, at least within GR, we only have the choice between associating a coordinate-dependent pseudo-tensor with the energy of the gravitational field, or to restrict our attention to asymptotically flat spacetimes. For a historic discussion of the debate between Einstein, Lorentz and Levi-Civita on the first possibility see Cattani and DeMaria [1993], for a recent discussion of the second possibility see Poisson [2004], especially section 4.3.

bination is restricted to natural kinds.[49]

On the other hand, the proposal that relational properties can be possessed essentially is not new either. Already Moore [1922] wrote:[50]

> Let P be a relational property, and a a term to which it does in fact belong. I propose to define what is meant by saying that P is internal to a (in the sense we are now concerned with) that from the proposition that a things [sic] has not got P, it 'follows' that it is *other* than a.

One famous candidate for an essential relational property, arising from an internal relation, has been proposed by Kripke , and it goes under the name 'the essentiality of origin'. Kripke argues that it is an essential relational property of every person that in every possible world they would have the same parents they *actually* have (Kripke [1972], p.113):

> How could a person originating from different parents, from a totally different sperm and egg, be *this very woman*? One can imagine, *given* the woman, that various things in her life could have changed [...]. But what is harder to imagine is her being born of different parents. It seems to me that anything coming from a different origin would not be this object.

Let us accept Kripke's assertion of the essentiality of origin as a working hypothesis for now. Of course, there is an asymmetry here: even if one thinks that it is an essential relational property of *Albert* that he is Hermann Einstein's son, this does not necessarily mean that it is also an essential

[49]Cf. Lewis [2001]. The *principle of recombination* states that it is possible to recombine the natural kinds (like horses, horns, electrons) of the *actual* world in order to get a new *possible* world. The *principle of plenitude* states that there actually *is* a possible world for *every* recombination of the natural kinds of our world. The principle of recombination then implies that a relation that holds in our world must not hold in every possible world (there is a possible world in which Hermann Einstein does not have a son), and the principle of plenitude implies that for every recombination there is indeed a possible word arising from it. So there is no relation, and hence no relational property, that holds true of the same object in every possible world. Hence, both principles together with our definition of what an essential property of an object is (a property the respective object *could not* lack) imply that every essential property *must* be an intrinsic property.

[50]Dunn [1990] distinguishes different senses of 'internal relation' with the help of his relevance logic, but I will not introduce them here.

relational property of *Hermann* that he is Albert's father. In another world, Hermann may not have had children, and still be the same Hermann Einstein, whereas, according to Kripke, it is not possible for Albert to have another father than Hermann and still be the same Albert Einstein.

Anyhow, note that the claim that mass-energy-momentum is an essential relational property of matter is a much weaker claim than Kripke's. There is a crucial difference between claiming that every person essentially has a *particular* pair of people as parents, and claiming that every material system essentially possesses *some* mass-energy-momentum density.

So our case corresponds to a weaker version of Kripke's original claim, which one could formulate as follows: In order for Φ to be a *human* being, it needs to have *some* parents, but *not* a *particular* pair. Similarly, in order for Φ to be a *matter* field, it needs to have *some* mass-energy-momentum density tensor $T_{\mu\nu}$ associated with it. The structure of its energy tensor $T_{\mu\nu}$ only tells us what *kind* of matter a given matter field Φ can describe — for example, we need vanishing diagonal components for an electromagnetic field and non-vanishing diagonal components for a perfect fluid — but in order to know that it is a *matter* field in the first place, we only need to know that there is an energy tensor associated with the field.

5 Conclusion

I started out by describing how Newton's defining property of *material* systems — the property of possessing mass — was generalised by relativistic field theory to the requirement that such systems need to have a mass-energy-momentum density tensor $T_{\mu\nu}$ associated with them.

We have then seen that energy tensors depend on both the matter fields Φ and the metric field $g_{\mu\nu}$, and looked at how energy tensors are derived within the framework of Lagrangian field theories. We thereby saw that the existence of mass-energy-momentum depends on the matter fields being related to the metric field in a certain way: they need to *couple* to the metric field.

Hence, in relativistic field theory, mass-energy-momentum cannot be regarded as an intrinsic property of matter, but must be seen as a relational property of matter (or a property of systems containing matter) that it only has because of its relation to spacetime structure.

Is it a surprising result that $T_{\mu\nu}$ does not describe an intrinsic property of matter? After all, nobody would have claimed that the momentum of a particle in Newtonian mechanics was an intrinsic property of this particle; it is a property that only makes sense if we describe that particle as changing its position with respect to something else. Even (kinetic) energy might be regarded as a relational property in Newtonian mechanics, for it depends on the velocity of the particle, which is a relational property for the same reason as the momentum of the particle.

On the other hand, mass *is* an intrinsic property of material systems in Newtonian mechanics: a body's mass does not depend on the relation the body has to other bodies, and given that the mass-energy momentum density $T_{\mu\nu}$ in the Einstein equations takes over the role the mass density ρ_M played in the Poisson equation of Newtonian theory, it may well be seen as a surprise: the most important property of matter, and it is not intrinsic to it.

In the introduction, I mentioned that Einstein himself was strongly motivated by a version of Mach's principle when he created GR, a version in which he claimed the geometric field $g_{\mu\nu}$ should be determined by the energy tensor $T_{\mu\nu}$. Einstein's idea was that spacetime structure should be derived from the properties of material systems. We have seen that the energy tensor already depends on spacetime structure, and that hence even a unique determination of $g_{\mu\nu}$ by $T_{\mu\nu}$ would not be sufficient for spacetime geometry to be reduced to the properties of matter.

But neither does GR accord to what might be called an Anti-Machian principle: even though the matter fields Φ do not determine the spacetime structure $g_{\mu\nu}$, spacetime does not determine the material structures either: both sides only constrain each other. In order to get a truly Anti-Machian theory, we would need not only the energy tensor to depend on the metric field, but the matter fields themselves would need to be derivable from the structure of spacetime. An example of such a theory is Kaluza-Klein theory, in which the electromagnetic vector potential forms part of the 5-dimensional metric tensor, and hence leads to the electromagnetic field $F_{\mu\nu}$ itself to be derivable from the geometric properties of spacetime.

But this leads us too far afield. The energy-momentum tensor of matter depends on, but is not determined by, the structure of spacetime — and that is enough for today.

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