Is the Statistical Interpretation of Quantum Mechanics ψ-Ontic or ψ-Epistemic?

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The ontological models framework distinguishes ψ-ontic from ψ-epistemic wave-functions. It is, in general, quite straightforward to categorize the wave-function of a certain quantum theory. Nevertheless, there has been a debate about the ontological status of the wave-function in the statistical interpretation of quantum mechanics: is it ψ-epistemic and incomplete or ψ-ontic and complete? I will argue that the wave-function in this interpretation is best regarded as ψ-ontic *and incomplete*. Furthermore, I will show that the probabilities in the statistical interpretation also point to the incompleteness of the theory if construed as hypothetical frequencies.

Contents

1 Introduction

What is the correct ontology for quantum mechanics? Since textbook quantum mechanics is notoriously vague about its ontological commitments, philosophers have discussed three different completions of quantum mechanics: the de Broglie–Bohm pilot-wave theory, the GRW collapse theory, and the Everettian many-worlds theory. Each of these completions of quantum mechanics provides a different picture of the world and suggests different mechanisms underlying quantum phenomena. At most one of these theories can match the true structure of the world. Since all three theories are (largely) empirically equivalent, the question of the right ontology cannot be solely settled by empirical evidence.[1] Instead, one would need to argue for theoretical virtues, like simplicity (Esfeld et al., 2017), explanatory power (Maudlin, 2016), predictive power (Allori et al., 2014), empirical coherence (Barrett, 1996), empirical adequacy (Barrett, 2020), etc.

Another way to react to this type of underdetermination of the ontology of quantum mechanics is to distill the minimal ontological commitments of standard quantum mechanics that are empirically confirmed and that do not rely on speculations about an underlying metaphysics. One would then develop a view of scientific realism that commits you to the truth of this "empirically confirmed" ontology without committing you to a particular fundamental ontology as provided by the pilot-wave theory, the GRW theory, or the many-words theory (see Rosaler, 2016; Saatsi and French, 2020; Egg, 2021).

The statistical interpretation of quantum mechanics would be such a candidate theory to provide a minimal ontology of the empirical core of quantum mechanics. It interprets the wave-function as representing properties of ensembles instead of individual systems, and the probabilities in this theory are defined in terms of observable frequencies. In this sense, the statistical interpretation is close to empirical practice; it is minimal in its ontological commitments; and it is open to be completed by a fundamental ontology. The statistical interpretation gives, therefore, a much more straightforward picture of what quantum mechanics is about than textbook quantum mechanics, which is often presented as a mixture of operationalism and some form of Copenhagen interpretation.

I want to clarify one aspect of the ontology of the statistical interpretation, namely, the ontological status of the wave-function therein. Does it give a complete or incomplete description of physical reality? Two opposing answers have been given using the ontological models framework: Harrigan and Spekkens (2010) regard the wave-function as ψ-epistemic and incomplete, while Oldofredi and López (2020) regard it as ψ-ontic and complete. I will argue that both interpretations face different kinds of problems: the interpretation of Harrigan and Spekkens (2010) runs into the PBR-theorem, and the interpretation of Oldofredi and López (2020) is in fact incomplete itself and conceptually unclear. Ultimately, the status of the wave-function in the statistical interpretation depends on the particular completion of this theory. Yet, I will argue that the wave-function in the statistical interpretation is most plausibly ψ-ontic and incomplete.

[1] The GRW theory does make different predictions from the other two quantum theories (see, for instance, Sebens, 2015). So far it has not been empirically refuted. I want to put this aspect of the GRW theory to the side.

2 The Ontological Models Framework

In their recent article, Oldofredi and López (2020) claim that Harrigan and Spekkens (2010) define the ontic state λ in too restrictive a way so that it does not properly capture the statistical interpretation of quantum mechanics. Furthermore, they argue that Harrigan and Spekkens (2010) put the wave-function in the statistical interpretation in the wrong category: instead of being ψ-epistemic (and incomplete) the wave-function is ψ-ontic (and complete). They support their argument by historic evidence from Albert Einstein and Leslie Ballentine, who are the most prominent proponents of the statistical interpretation. I want to challenge all three claims and argue for the following: (i) the ontic state in the sense of Harrigan and Spekkens (2010) is general enough to capture a statistical interpretation of the wave-function, (ii) the wave-function is therefore indeed ψ-epistemic as perceived by Einstein and Ballentine (although it does not merely represent an observer's knowledge), and (iii) the statistical interpretation is more compatible with an ontic state as described by Harrigan and Spekkens (2010) than as re-defined by Oldofredi and López (2020).

2.1 What is the Ontological Models Framework?

Let me first explain where the notion of ψ-epistemic comes from and what it means. Harrigan and Spekkens (2010) introduce it within a certain formal framework for quantum mechanics, which they call the *ontological models framework*. The main question they ask is, "Does the quantum state represent reality or only our knowledge of reality?" One option is to sidestep this question and take an operational approach to quantum mechanics: all one is interested in is predicting the behavior of quantum systems without recourse to unobservable objects and processes. The ideal case would be an *operational quantum theory* that describes all predictions of the theory in terms of preparation and measurement procedures. In doing so, it needs to say what kinds of measurements M and preparation procedures P yield an outcome k. An operational quantum theory specifies a probability $\mathbb{P}$ for k, given M and P, that is, $\mathbb{P}(k|M,P)$.

Often a physical theory tells us more about the world than how certain manipulations on a physical system lead to certain empirical results: it may tell us what the measured system is made of, what a measurement device is made of, and how measurement and preparation devices interact with the constituents of the measured system (Hubert, 2021b). In particular, the physical theory should (ideally) specify a complete description of the system's properties. This complete description is often denoted λ. A theory that gives such an ontological story to operational quantum theory is an *ontological model of operational quantum theory*.

Again, an ontological model of an operational theory of quantum mechanics provides more information about the physical system beyond an operational level; it describes the complete physical state λ of the system. Although measurement and preparation devices can be described with their own ontic state λ in an ontological theory, they are not reduced to these λ's in the ontological models framework. The main problem that the ontological models framework aims to tackle is how much an observer can know about the ontic state of the physical system if she prepares and measures the system in a certain way. For example, a preparation procedure may not uniquely fix the ontic state but rather a probability distribution $\mathbb{P}(\lambda|P)$ for ontic states given a preparation procedure. If the observer prepares the system in a state λ, this ontic state determines then the result k once measured, that is, the ontological models framework provides this probability distribution $\mathbb{P}(k|\lambda, M)$. We can now explain the predictions of the operational theory $\mathbb{P}(k|M, P)$ with the machinery of the ontological theory:

$$\mathbb{P}(k|M, P) = \int \mathrm{d}\lambda\, \mathbb{P}(k|\lambda, M)\mathbb{P}(\lambda|P).$$

We integrate over all the possible ontic states, where $\mathbb{P}(\lambda|P) \neq 0$ singles out the ontic states that are compatible with the state procedure P.[2]

Oldofredi and López (2020) seem to mischaracterize the relation between an operational theory and its corresponding ontological model:

> It is worth noting that the authors define ontological models employing an operational setting, i.e. the primitive notions of such models consist exclusively in preparations procedures of physical systems in certain states and measurements performed on them. A complete specification of the properties of a given physical system is provided by λ, the ontic state of the system under scrutiny. (Oldofredi and López, 2020, pp. 1318–1319)
>
> As we have seen, ontological models consider operational procedures to prepare the state of a quantum system in a certain manner as primitive notions. Such procedures are associated with some observable properties, whose values will be then revealed by the performance of a set of measurements on the physical system under scrutiny. (Oldofredi and López, 2020, pp. 1321)

It is not the ontological model that treats the operational procedures as primitive notions. Rather, the ontological model provides the tools for *explaining* these procedures and *reduce* them to the behavior of the ontic state λ. It is only on the level of the operational quantum theory, where the operational procedures are taken as primitive. Therefore, the relation between the operational theory and the corresponding ontological model is indeed similar to the relation between thermodynamics and statistical mechanics.[3]

In quantum mechanics, the wave-function has a double role: it determines the probabilities of measurement outcomes via the Born rule, and it also (at least partially) describes the ontic state of the physical system. This is also reflected in the ontological models framework. The probabilities

[2]It is often implied that the ontological model framework is a strong constraint on a physical theory (for instance, Gao, 2017). It is rather a formalization of how physics has been done (until recently with the development of quantum physics) by identifying the complete physical state of a system and investigating the time evolution of this state.

[3]Harrigan and Spekkens say, "In an ontological model of an operational theory, the primitives of description are the properties of microscopic systems." (p. 128) The formalism of the ontological models framework, however, would be also general enough to incorporate primitives of description that are not strictly speaking "microscopic": for example, wave-function realism would be a viable candidate to underpin operational quantum theory. I thank Alyssa Ney for this insight.

of the operational theory are determined by the wave-function $\mathbb{P}(k|M,P) = \mathbb{P}_\psi(k|M,P)$. The ontological models framework zooms in on the relation between the ontic state λ and the corresponding wave-function describing this state. If the system is prepared to have a certain wave-function ψ, the system may be in one of many possible ontic states compatible with this ψ. The corresponding probability distribution would be $\mathbb{P}(\lambda|P = \psi)$.

Having introduced the ontic state λ, one can then try to answer the original question "Does the quantum state represent reality or our knowledge of reality?". In other words, does the wave-function represent objective properties of the ontic state (the complete physical description of the system), or does it rather represent an agent's knowledge about properties of this state?

Harrigan and Spekkens (2010) introduce an ingenious and rather general formal distinction to make this question more precise. If we prepare two systems with different wave-functions, a natural question arises of how the ontic states corresponding to these wave-functions are related. The ontological models framework distinguishes between two cases:

(i) If the probability distributions $\mathbb{P}(\lambda|P = \psi_A)$ and $\mathbb{P}(\lambda|P = \psi_B)$ of two different wave-functions ψ_A and ψ_B do not overlap, as depicted in Fig. 1a), the wave-function is called *ψ-ontic*.

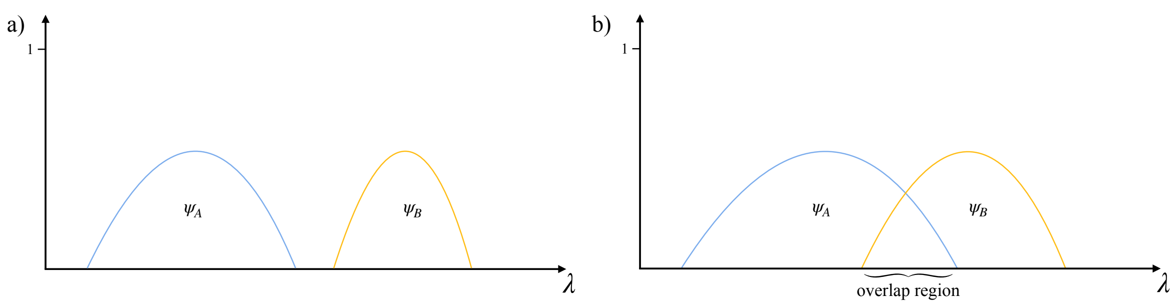

Figure 1: a) The definition of a ψ-ontic wave-function: The probability distributions over the ontic states associated with the wave-functions ψ_A and ψ_B are disjoint. b) The definition of a ψ-epistemic wave-function: The probability distributions over the ontic states associated with the wave-function ψ_A and ψ_B share a region of overlap

(ii) If the probability distributions $\mathbb{P}(\lambda|P = \psi_A)$ and $\mathbb{P}(\lambda|P = \psi_B)$ of two different wave-functions ψ_A and ψ_B do overlap, as depicted in Fig. 1b), the wave-function is called *ψ-epistemic*.

For every wave-function, there is a set of ontic states λ compatible with this wave-function, that is, there is a one-to-many relation from the wave-function to the underlying ontic states. If we pick an ontic state, then it depends on the wave-function whether the relation from the ontic state to the wave-function is one-to-one or one-to-many. More precisely, if the wave-function is ψ-ontic, fixing an ontic state λ, there is only one wave-function associated with it. If the wave-function is

ψ-epistemic, there are ontic states which are associated with at least two wave-functions—namely, the ones in the overlap region.

The definition of ψ-ontic and ψ-epistemic was up to now only about whether or not a wave-function can be uniquely associated with an ontic state. The definitions have been in fact made to clarify in what way the wave-function is ontic or epistemic. But we will shortly encounter certain caveats regarding the ontological and epistemic status of ψ-ontic and ψ-epistemic wave-functions.

2.2 How Ontological Are ψ-Ontic Wave-Functions?

Let's first discuss in which way a ψ-ontic wave-function is ontological. If an ontic state is uniquely associated with a wave-function, then and only then can we interpret the wave-function as representing only certain objective properties of λ (what these properties exactly are is left open).

We can introduce further definitions that show how a ψ-ontic wave-function can objectively represent these properties (see Fig. 2a). First, the wave-function alone can completely describe the state of the system. In this case, we call the wave-function ψ-complete. The probability distributions of ψ-complete wave-functions are sharply peaked around λ, as shown in Fig. 3. If the wave-

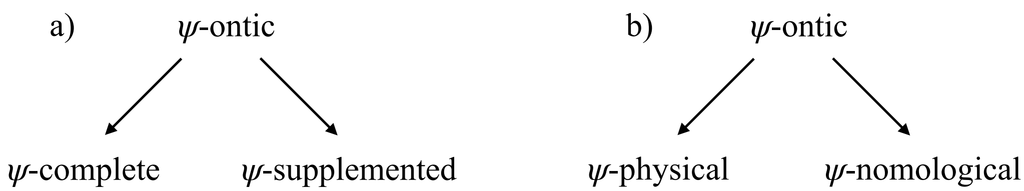

Figure 2: a) ψ-ontic wave-function can be either ψ-complete or ψ-supplemented. A ψ-complete wave-function describes completely the ontic state λ. A ψ-supplemented wave-function only gives a partial description. b) A ψ-ontic wave-function can be either a physical object (ψ-physical) or an abstract object representing properties of the system (ψ-nomological).

function does not completely describe λ, then it is called ψ-supplemented. A ψ-supplemented wave-function would need additional variables to provide such a complete description (what these variables are is also left open in the ontological models framework).

We can also describe ψ-ontic wave-functions in a different way, as done in Fig. 2b). If the wave-function represents properties of λ because it is itself a physical object (or rather the object it represents is a physical object), then we call it ψ-physical. In John Bell's terminology, a ψ-physical wave-function would be a beable. If the wave-function represents an abstract (non-physical) entity that still determines the behavior of λ, we call it ψ-nomological.

We can now build a matrix for the different combinations of ψ-ontic wave-functions (see Fig. 4), as the ontological models framework merely provides a general categorization of the wave-function that needs to be filled in by specific interpretations of quantum mechanics. Example of theories or interpretations that construe the wave-function as ψ-complete and ψ-physical are wave-function realism (Ney, 2021), probably the Copenhagen interpretation, and different versions of

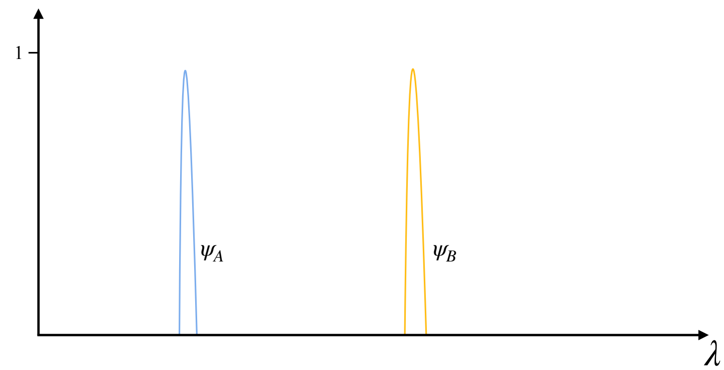

Figure 3: ψ-complete wave-function. The wave-functions ψ_A and ψ_B are narrowly peaked around an ontic state λ.

the many-worlds interpretation (Carroll, 2019). Examples of ψ-supplemented and ψ-physical are the multi-field interpretation (Hubert and Romano, 2018; Romano, 2021), space-time state realism in the many-worlds theory (Wallace, 2003), the many-worlds theory with matter density (Allori et al., 2011), and the original version of Albert's marvelous point interpretation (Albert, 1996). Examples of theories or interpretations with a ψ-supplemented and ψ-nomological wave-function would be Humean interpretations of the de Broglie–Bohm theory (also called Bohumianism, Esfeld et al., 2014, 2017; Miller, 2014; Callender, 2015; Bhogal and Perry, 2017; Dewar, 2020), the dispositional interpretation of the wave-function in the de Broglie–Bohm theory (Esfeld et al., 2014, 2017; Suárez, 2015), the wave-function as a nomological entity in the de Broglie–Bohm theory (Goldstein and Zanghì, 2013), wave-functionalism (Allori, 2021), a Humean version of Albert's marvelous point interpretation (Loewer, 1996), and the GRW theory with a flash or matter ontology (Dorato and Esfeld, 2010; Egg and Esfeld, 2015; Lorenzetti, 2021).[4] A ψ-nomological interpretation with a ψ-complete wave-function has not been proposed, since it is hard to grasp how the wave-function can completely represent the ontic state with it being some non-physical entity.

2.3 How Epistemic Are ψ-Epistemic Wave-Functions?

Now let us turn to ψ-epistemic wave-functions. These are the ones where some ontic states λ are associated with more than one wave-function. How can that happen? One obvious way is when two agents have different knowledge about the same system and disagree on the the wave-function, as in Fig. 5. It would be a mistake to say that the wave-function in this case is purely epistemic; rather, agents would learn something about the system if they assign to the system one of the (correct) wave-functions that are associated with the ontic state.[5] This kind of ψ-epistemic

[4]The philosophical literature on the GRW theories seems to imply that the universal wave-function is ψ-nomological. It still needs to be worked out what a ψ-physical wave-function amounts to for collapse theories.

[5]I thank Travis Norsen for this insight.

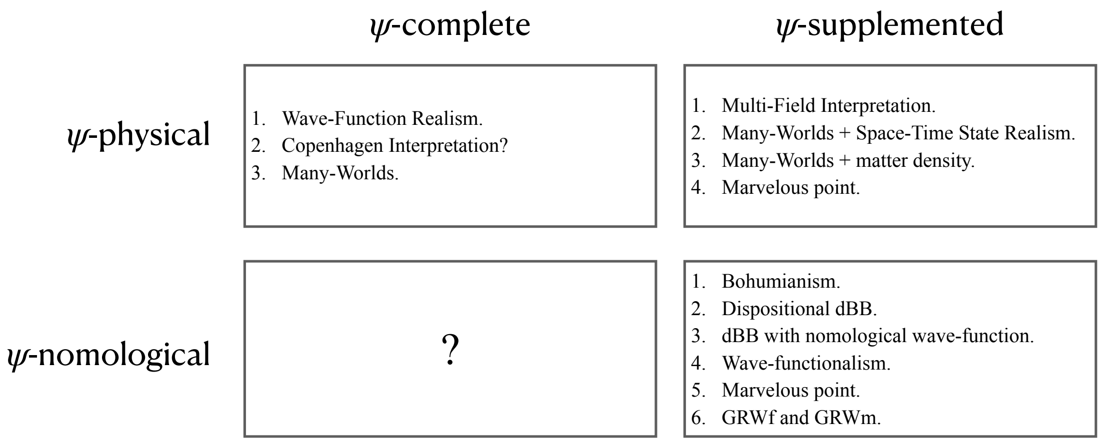

Figure 4: The many examples of a ψ-ontic wave-function.

wave-function emphasizes the relational character between the objective properties of λ and the agents' knowledge about λ represented in the wave-function. We could call such a wave-function ψ-credal, as there is another way to interpret ψ-epistemic wave-functions (see Maudlin, 2019, Ch. 3).

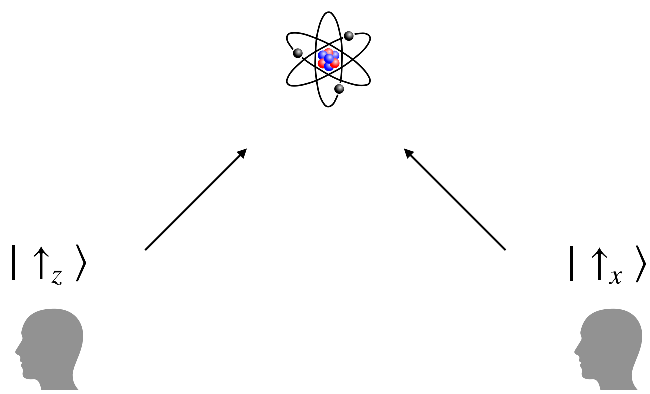

Figure 5: One possibility of a ψ-epistemic wave-function: Two agents assign two different wave-functions to the same system.

Oldofredi and López (2020) assume that *all* ψ-epistemic wave-functions are ψ-credal:

> If a model is ψ-epistemic, then it cannot in any case be ψ-ontic, since it does not describe any underlying physical reality, but only the agents' knowledge of it. (Oldofredi

and López, 2020, pp. 1320)[6]

Even if a ψ-epistemic wave-function were to represent an agent's knowledge, the wave-function would be referring to the ontic state. So if I assign a correct ψ-epistemic wave-function to a quantum system, I would indeed know some aspects of the underlying physical reality, even if another agent may assign a different wave-function to the same system.

It is indeed possible to interpret a ψ-epistemic wave-function in a non-epistemic way.[7] Imagine we prepare two beams of particles, one with wave-function ψ_A and the other with wave-function ψ_B, and we assume that the wave-function describes ensembles. Then whether a particle has wave-function ψ_A is primarily a matter of whether it is part of the ensemble being prepared as having ψ_A as its wave-function. If the wave-function is ψ-epistemic there are some of the particles in the ψ_A ensemble that may be also correctly described by another wave-function, say ψ_B. This means that these particles can be correctly associated with either beam, the ψ_A beam or the ψ_B beam. In this case, the wave-function, although ψ-epistemic, does not represent an agent's knowledge but rather whether a particle is part of a certain ensemble. Maudlin (2019, Ch. 3) calls such a wave-function *ψ-statistical* (see Fig. 6).

This counterexample challenges two arguments by Oldofredi and López (2020): (i) that a ψ-epistemic wave-function is necessarily epistemic, and (ii) that the wave-function in the statistical interpretation has to be ψ-ontic. Having replied to (i), I will discuss (ii) in section 4, but before doing so, I shall introduce the PBR-theorem, which will be important to answer more completely whether the statistical interpretation of quantum mechanics can be ψ-epistemic. This will in particular affect how Harrigan and Spekkens (2010) understand the statistical interpretation.

3 The PBR-Theorem

The PBR-theorem starts with two crucial assumptions:

- **Assumption 1 (Reality Criterion)**: Every physical system has an underlying ontic state that completely describes its physical properties, which is objective and independent from observers.

- **Assumption 2 (Preparation Independence)**: Two preparation devices run independently from each other.

The first assumption is essential to the ontological models framework, whereas the second is added for methodological reasons. The reality criterion basically says that we consider a theory that fits

[6] They say the same also on p. 1318 and 1328.

[7] It is also possible to have an epistemic interpretation of the wave-function without it being ψ-epistemic. Therefore, the distinction between ψ-ontic and ψ-epistemic wave-functions does not cover all the ways one can interpret the wave-function. Pace Oldofredi and López (2020, pp. 1320), the wave-function in QBism would be such an example, as this theory denies the existence of λ, which is necessary for a wave-function to be ψ-epistemic. What the ontology of QBism actually is is still debated (see, for instance, Boge, 2021). On the other hand, Oldofredi and López (2020, p. 1341) classify relational quantum mechanics to be ψ-epistemic because: "For RQM, by contrast, the quantum state is merely a useful tool for calculation and prediction, and because of this it is ψ-epistemic."

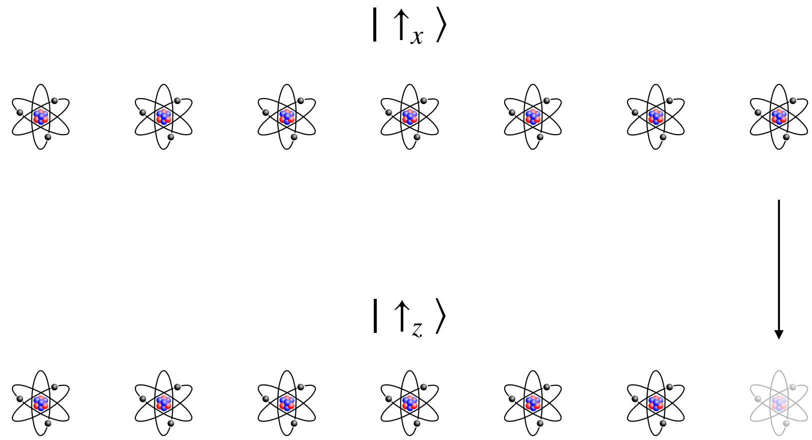

Figure 6: The definition of a ψ-statistical wave-function: The wave-function describes ensembles of equally prepared quantum systems. In the picture, the upper beam of quantum systems is prepared in a spin x-up state; the lower beam in a spin z-up beam. By definition of ψ-epistemic, a certain fraction of quantum systems in each beam can be described by two wave-functions. Therefore, say, the last system in the upper beam can be also correctly associated with the spin z-up ensemble.

into the ontological models framework. Preparation independence adds to the ontological models framework that the statistical distributions of ontic states that are generated in one preparation device do not depend on the statistical distributions of ontic states that are generated in another preparation device.[8]

The general argument of the PBR-theorem is the following: if the wave-function is ψ-epistemic and the two assumptions hold, then we get a contradiction with the (well-confirmed) predictions of quantum mechanics. Since the two assumptions are reasonable, the wave-function cannot be ψ-epistemic and must be therefore ψ-ontic. It is indeed possible to question the validity of the two assumptions and explore how such a quantum theory would look like.[9] I just want to emphasize here that it is impossible to violate the Reality Criterion and retain a ψ-epistemic wave-function, as is often claimed. For the definition of ψ-epistemic hinges on the ontological models framework and thus on the existence of λ.

[8] One may reason that the preparation devices need to be space-like separated to run independently. That is just one way to justify that they have to run independently. Even if the preparation devices are not space-like separated assumption 2 may still hold if there is no causal relation between the devices.

[9] Theories that deny the reality assumption are QBism (Fuchs, 2017; Fuchs and Schack, 2014), radical epistemicism (Ben-Menahem, 2017, 2018, 2020), relational quantum mechanics (Rovelli, 1996; Oldofredi and López, 2020; Oldofredi and Calosi, 2021; Di Biagio and Rovelli, 2021), and pragmatist interpretations of quantum mechanics (Healey, 2017). Theories that deny preparation independence are Spekken's toy model (Spekkens, 2007) and different proposals for superdeterministic theories (Palmer, 1995; 't Hooft, 2016; Hossenfelder and Palmer, 2019; Ciepielewski et al., 2021). An excellent critical review of this approach is Chen (2021).

Let us now discuss the experimental set-up of the PBR-theorem. Two preparation devices are used that generate particles either in a z-spin up or an x-spin up state (see Fig. 7). Alice and Bob can

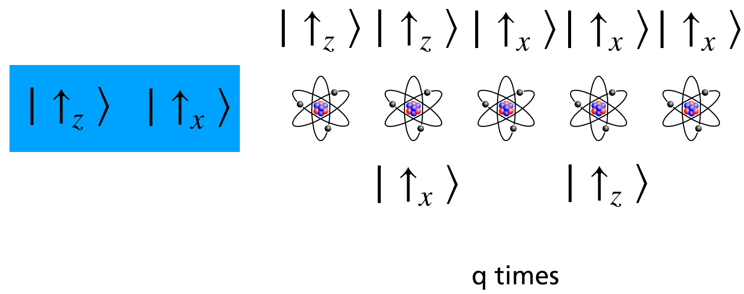

Figure 7: The preparation of systems in the PBR-theorem. If the wave-function is ψ-epistemic, some of the prepared system can are associated with two different wave-functions. It is assumed that this happens q times.

choose in which state the system should be prepared. Since it is assumed that the wave-function is ψ-epistemic some of the wave-functions in Alice's ensemble can be correctly associated with another wave-function. For simplicity, we assume that if such a system is in an z-spin up state the other correct wave-function would be x-spin up and vice versa. The same is the case for Bob's systems. To have a concrete number, we say that these ontic states with double wave-functions appear q times in such an ensemble.

Now Alice and Bob shoot their quantum systems into a measuring device that is set up to measure pairs of particles, where one system of the pair comes from Alice and the other from Bob. Certain pairs are of particular interest, because these are the ones that ultimately yield a contradiction with the predictions of quantum mechanics. A fraction of q systems in Alice's and on Bob's side are associated with two wave-function: $|\uparrow_z\rangle$ and $|\uparrow_x\rangle$. Due to preparation independence, the ontic states with such double wave-functions on Alice's side are independent from (or uncorrelated with) the ontic states with double wave-functions on Bob's side. Therefore, q^2 times a pair of particles is associated with four wave-functions (each in a product state): $|\uparrow_z\rangle|\uparrow_z\rangle$, $|\uparrow_z\rangle|\uparrow_x\rangle$, $|\uparrow_x\rangle|\uparrow_z\rangle$, and $|\uparrow_x\rangle|\uparrow_x\rangle$.

The subsequent measurement is set up in such a way to yield one of four possible measurement outcomes: A, B, C, or D. What are the probabilities for either outcome if a pair in a product state is measured? The device is prepared that a $|\uparrow_z\rangle|\uparrow_z\rangle$ state will not get measured as A, that is, the probability for yielding A given this product state is zero: $P_{z,z}(A) = 0$. Similarly, for the other product states: $P_{z,x}(B) = 0$, $P_{x,z}(C) = 0$, $P_{x,x}(D) = 0$.[10] It is now easy to see why these ontic states that are associated with all these product states are problematic. If we shoot these states into the measurement device, the theory tells us that the measurement will not give us either of the four possible results. On the other hand, the measurement apparatus by construction

[10]For our purposes it is not important to explain the details of this measurement procedure. The product states get projected onto a certain entanglement basis that gives rise to these probabilities.

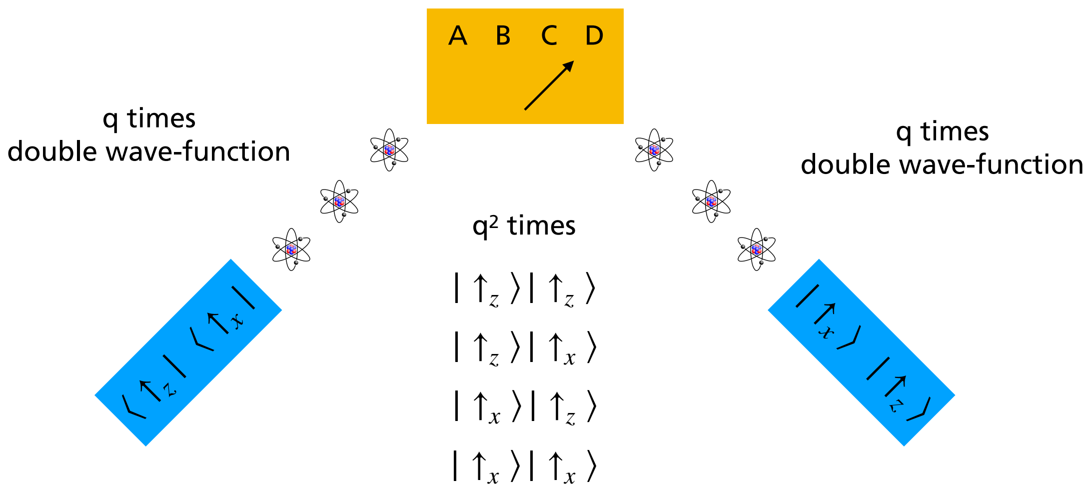

Figure 8: The proof of the PBR-theorem. Alice and Bob prepare an ensemble of quantum system that are either in a z-spin up or and x-spin up state. They send their system to a measurement device (orange box) that makes a measurement on pairs of systems, one system in such a pair is from Alice's and one from Bob's preparation device.

will measure something and will yield one of the four possible outcomes.[11] Overall, it happens q^2 times that the measurement device will show a result that is prohibited by quantum mechanics if the wave-function is ψ-epistemic.

One can now react in three different ways to the result of the PBR-theorem:

1. Our best choice of wave-functions is to interpret them as ψ-ontic.
2. We may deny the preparation independence assumption. These theories would fit into the ontological models framework, they would also have a ψ-epistemic wave-function, but they would not lead to the contradiction indicated in the PBR-theorem.[12]
3. We may deny the reality criterion. Then the entire set up of the ontological models framework would be undermined; in particular, the PBR-theorem would say nothing about these

[11] The precise argument why the measurement apparatus has to show a result is a bit technical. Roughly speaking it goes like this. We deal here with a four-dimensional Hilbert space. Each outcome of the measurement device is associated with a basis vector in this Hilbert space. These basis vectors span the *complete* Hilbert space. So upon measurement any vector in this Hilbert space gets projected on one of these basis vectors and yields the corresponding value with a certain probability. In particular, the product states $|\uparrow_z\rangle|\uparrow_z\rangle$, $|\uparrow_z\rangle|\uparrow_x\rangle$, $|\uparrow_x\rangle|\uparrow_z\rangle$, and $|\uparrow_x\rangle|\uparrow_x\rangle$ are part of this Hilbert space. So it is impossible that the measurement device will yield nothing if fed with one of the problematic ontic states.

[12] It is, in principle, a further option to have a ψ-ontic wave-function and deny preparation independence. Ciepielewski et al. (2021) present such a model, although they do not defend it.

kinds of quantum theories.

I do not want to argue for or against either of these strategies in this paper. Instead, this overview will give us a scheme for evaluating the status of the wave-function in the statistical interpretation of quantum mechanics.

4 The Statistical Interpretation of Quantum Mechanics

For Harrigan and Spekkens (2010), the wave-function in the statistical interpretation is ψ-epistemic and ψ-incomplete, while Oldofredi and López (2020) claim it to be ψ-ontic and ψ-complete. I will argue that Einstein and Ballentine regard the wave-function in the statistical interpretation as ψ-epistemic and ψ-incomplete. I will also discuss whether the wave-function in the statistical interpretation can be ψ-statistical. I will conclude, however, that the wave-function in the statistical interpretation needs to be ψ-ontic and ψ-incomplete.

Let's first start to introduce the statistical interpretation of quantum mechanics. One of the clearest descriptions of it can be found in Einstein's 1949 *Reply to Critics* in Paul Schilpp's volume *Albert Einstein: Philosopher–Scientist*:

> Within the framework of statistical quantum theory there is no such thing as a complete description of the individual system. More cautiously it might be put as follows: The attempt to conceive the quantum-theoretical description as the complete description of the individual systems leads to unnatural theoretical interpretations, which become immediately unnecessary if one accepts the interpretation that the description refers to ensembles of systems and not to individual systems. In that case the whole 'egg-walking' performed in order to avoid the 'physically real' becomes superfluous. There exists, however, a simple psychological reason for the fact that this most nearly obvious interpretation is being shunned. For if the statistical quantum theory does not pretend to describe the individual system (and its development in time) completely, it appears unavoidable to look elsewhere for a complete description of the individual system; in doing so it would be clear from the very beginning that the elements of such a description are not contained within the conceptual scheme of the statistical quantum theory. With this one would admit that, in principle, this scheme could not serve as the basis of theoretical physics. Assuming the success of efforts to accomplish a complete physical description, the statistical quantum theory would, within the framework of future physics, take an approximately analogous position to the statistical mechanics within the framework of classical mechanics. I am rather firmly convinced that the development of theoretical physics will be of this type; but the path will be lengthy and difficult. (Schilpp, 1949/1970, pp. 671–672)

I want to take home three important points from this passage:

1. The wave-function in the statistical interpretation describes ensembles instead of individual quantum systems.

2. Therefore individual systems are not completely described by the wave-function.
3. To describe an individual system completely, one would need to go beyond the statistical interpretation to search for such a completion.

The statistical interpretation, as Einstein describes it, is a peculiar form of interpretation. Normally, one would seek an interpretation of quantum mechanics that is in some sense complete. Even if one were to support an operational or pragmatist interpretation of quantum mechanics (Healey, 2017), one would argue that such an interpretation does not need a completion—for example, because one is skeptical that we can discover the true nature of unobservable objects. The statistical interpretation, on the other hand, is more like a half-interpretation: for all practical purposes, one can think of the wave-function describing ensembles, but this is not the complete story of reality. In his last words on the statistical interpretation, Einstein emphasized in a letter to A. Lamouche just a month before he died on March 20, 1955, that the statistical interpretation is incomplete:

> The ψ-function is not to be considered as a complete description of an individual state of affairs, rather only as a representation of what we can know about a particular state of affairs from an empirical point of view. Then the ψ-function is a representation of an "ensemble", not the complete characterization of individual states of affairs. One has thereby renounced the latter in principle.[13] (quoted in Fine, 1993, p. 9)

We can find the same attitude in Ballentine (1970, 1972, 2015), who is probably the most famous modern advocate of the statistical interpretation:

> The Statistical Interpretation, according to which a pure state provides a description of certain statistical properties of an ensemble of similarly prepared systems, but need not provide a complete description of an individual system. (Ballentine, 1970, p. 360)

> We see that a quantum state is a mathematical representation of the result of a certain state preparation procedure. Physical systems that have been subjected to the same state preparation will be similar in some of their properties, but not in all of them [. . .]. (Ballentine, 1970, p. 361)

> The Statistical Interpretation, which regards quantum states as being descriptive of ensembles of similarily prepared systems, is completely open with respect to hidden variables. It does not demand them, but it makes the search for them entirely reasonable [(]this was the attitude of Einstein (1949)[)]. (Ballentine, 1970, p. 374)

> The Statistical Interpretation does not prejudice the possibility of introducing hidden variables which would determine (in principle) the outcome of each individual measurement (Sec. 6). (Ballentine, 1970, p. 379)

[13] I thank Maaneli Derakhshani for this reference.

For both Einstein and Ballentine, the wave-function does indeed give an incomplete description of physical systems and would need a completion to provide such a description—although Ballentine's statement that "The Statistical Interpretation does not prejudice the possibility of introducing hidden variables" is a bit mysterious if read together with his statement that the statistical interpretation "is completely open with respect to hidden variables". On the one hand, it seems that Ballentine regards the statistical interpretation as a viable interpretation on its own right, but, on the other hand, this interpretation is incomplete and can be completed by hidden variables. A charitable reading would dissolve this tension: the statistical interpretation is a useful interpretation of quantum mechanics, especially for physicists, and it gives a clear statistical interpretation of the predictions of quantum mechanics. This interpretation does therefore provide a richer picture of the world than pure operationalism and a less obscure metaphysics than the Copenhagen interpretation, even if the wave-function is ψ-incomplete.

4.1 ψ-epistemic and incomplete?

Is the wave-function in this interpretation ψ-ontic or ψ-epistemic? Einstein does not explicitly answer this question in the above quotation. But since the wave-function describes ensembles instead of individual quantum systems, it seems plausible that an individual system can be associated with two different wave-functions. Einstein mentions such a case in his 1935 correspondence with Schrödinger (see Howard, 1985, section 2 and Harrigan and Spekkens, 2010, section 4.3):

> Now what is essential is exclusively that ψ_B and $\psi_{\underline{B}}$ are in general different from one another. I assert that this difference is incompatible with the hypothesis that the ψ description is correlated one-to-one with the physical reality (the real state). After the collision, the real state of (AB) consists precisely of the real state of A and the real state of B, which two states have nothing to do with one another. *The real state of B thus cannot depend upon the kind of measurement I carry out on A* [...] But then for the same state of B there are two (in general arbitrarily many) equally justified ψ_B, which contradicts the hypothesis of a one-to-one or complete description of the real states. (Einstein to Schrödinger in 1935, quoted in Howard, 1985, p. 180)

Einstein concocted several arguments throughout his career to prove that quantum mechanics is incomplete. The only option Einstein saw in making sense of quantum mechanics was to interpret it as an incomplete statistical theory. So when Einstein concludes that quantum mechanics is incomplete, we can also understand this to say that the statistical interpretation is incomplete. His arguments also show what this incompleteness amounts to. In the above quote, Einstein talks about an entangled two-particle system AB, presumably in this state: $\psi = \frac{1}{\sqrt{2}}\Big(\psi_A\psi_B + \psi_{\underline{A}}\psi_{\underline{B}}\Big)$. We can think of two electrons in the singlet state, for example. Einstein presupposed that each particle is prepared in an ontic state λ_A and λ_B respectively. Whether or not we measure particle A, this measurement cannot have any physical influence on the other particle, since Einstein strongly believed in locality. Therefore, the ontic state of particle B remains the same before and after measurement, and the wave-function representing this state cannot change either. Thus, the ontic state λ_B is correctly described by both ψ_B and $\psi_{\underline{B}}$. Thus quantum mechanics, as well as the statistical interpretation, is ψ-epistemic and ψ-incomplete. Einstein deduces the incompleteness from the issue that several wave-functions can be associated with the same ontic state; a complete theory, on the other hand, would uniquely associate a wave-function to an ontic state. Harrigan and Spekkens also interpret this passage of Einstein's as arguing for a ψ-epistemic interpretation of the wave-function. They conclude:

> By characterizing his 1935 argument as one that merely established the *incompleteness* of quantum theory on the assumption of locality, Einstein did it a great disservice. For in isolation, a call for the *completion* of quantum theory would naturally have led many to pursue hidden variable theories that interpreted the fundamental mathematical object of the theory, the wave function, in the same manner in which the fundamental object of other physical theories were customarily treated—as ontic. But such a strategy was known by Einstein to be unable to preserve locality. Thus it is likely that the force of Einstein's 1935 argument from locality to the epistemic interpretation of ψ was not felt simply because the argument was not sufficiently well articulated. (Harrigan and Spekkens, 2010, p. 152)

Ballentine's answer as to whether the statistical interpretation is ψ-epistemic or ψ-ontic is encrypted in his second quote: "Physical systems that have been subjected to the same state preparation will be similar in some of their properties, but not in all of them." This sounds like he regards the wave-function as ψ-statistical. An individual system has a certain wave-function solely in virtue of being part of an ensemble that has been prepared in the same quantum state, *but some of these systems within this ensemble may differ with respect to their physical properties.* If these systems can be described by another wave-function, they would have a ψ-statistical wave-function, and then the wave-function would be ψ-epistemic.

If Ballentine would advocate such a statistical interpretation, it would not be able to make correct empirical predictions according to the PBR theorem. The only way out would be a ψ-epistemic completion of the statistical interpretation that violates the assumption of preparation independence, which would then lead to some form of super-determinism (see Spekkens, 2007; Leifer, 2014, for such a model).

Let me briefly explain why one needs to violate preparation independence to rescue ψ-epistemic wave-functions. Remember the PBR setting in Fig. 8 from section 3. Two preparation devices (blue boxes) send a beam of particles to a measurement device (orange box). The particles are either prepared in a spin x-up or spin z-up state. Since it is assumed that the wave-function is ψ-epistemic, a fraction q of the particles in each beam are described by both the spin x-up and spin z-up wave-functions (double wave-functions). Due to preparation independence, the ontic states of the left beam are independently distributed from the ontic states in the right beam; therefore, q^2 of times, a pair of particles (one from the left and one from the right beam) are associated with *four* wave-functions. These simultaneous four wave-functions lead to a contradiction with the predictions of quantum mechanics. A violation of preparation independence would not yield any two-particle system that is correctly described by these four wave-functions. That is, whenever a particle in the left beam has a double wave-function the corresponding particle in the right beam will only have one correct wave-function, and vice versa. Thus, the distribution of double wave-functions in each beam are correlated such that a particle with a double wave-function in one beam will not be paired with a particle in the other beam having also a double wave-function.

There are a couple of problems with violating preparation independence. First, how could the ontic states of the two beams be correlated in the first place? This becomes particularly problematic when we isolate each preparation device from each other—for example, by space-like separation and special isolation materials round them. There are two ways to do that, which are both implausible. Either these correlations happen by pure chance. If that were the case, there needs to be some case (even only in principle) where chance is not in our favor and would match two systems with a double wave-function. This would, however, undermine the empirical predictions of the theory. Another way to explain these correlations would be to postulate special fine-tuned initial conditions in the past before the preparation (and the measurement) were conducted. These fine-tuned initial conditions are ultimately traced back to fine-tuned initial conditions of the universe... and these fine-tuned initial conditions demand further explanation. Especially, when experimentalists do not do anything particularly special with the preparation devices, such special initial conditions have a mysterious character.

Second, the correlation of the ontic states after preparation is sensitive to the future measurement. If we change the measurement device and conduct a different kind of measurement, this would necessitate that the particles become differently correlated if this measurement is similarly prepared as in the PBR-theorem—otherwise, we would run to another contradiction. One may explain the influence of the measurement device again by special initial conditions, but this would make the set-up even more fine-tuned. And if the future measurement is not specially designed as in the PBR-theorem, does this mean that the particles would be still correlated or are they uncorrelated?

Third, connected to the previous point, a violation of preparation independence depends on operational procedures, like measurement and preparation. It is unclear where to draw the line between measurement and non-measurement procedures and between preparation and non-preparation procedures. It is therefore ill-defined in which situations one has correlations and in which one does not. Peter Lewis (2006) calls this a measurement-problem-like problem.

If the statistical interpretation is ψ-epistemic and incomplete as claimed by Harrigan and Spekkens (2010), it has to violate preparation independence and rely on some super-deterministic mechanism, otherwise it would not be an empirically adequate theory. To avoid a violation of preparation independence, you may seek an interpretation of the wave-function as ψ-ontic. Oldofredi and López (2020) go along this route but for different reasons. I present and evaluate their arguments in the next subsection.

4.2 ψ-ontic and complete?

The biggest problem Oldofredi and López (2020) identify in Harrigan and Spekkens's classification of the statistical interpretation is that this interpretation demands a different kind of ontic state:

> In the second place, another crucial point to highlight is that the ontic space of the

> statistical interpretation is not one of individuals, but of ensembles. This allows for an alternative reading of the ontic state: it provides a complete description of the properties of an ensemble, not of individuals. And there is nothing else to know about ensembles that is not provided by the quantum state. The upshot of the present discussion is that the sort of λ that the statistical interpretation poses is completely different in nature with respect to that employed by Harrigan and Spekkens. (Oldofredi and López, 2020, p. 1330)

According to Oldofredi and López (2020), the ontic state that underlies the statistical interpretation is not the one that is presupposed in the ontological models framework (which refers to individual systems), but one that only refers to an ensemble of systems. With this kind of ensemble-λ, one may then interpret the wave-function in the statistical interpretation to be ψ-ontic and also ψ-complete. Since, in the usual definition, a ψ-ontic wave-function requires an ontic state for individual systems, Oldofredi and López (2020) would need an "ensemble ontological model", in which the wave-function uniquely refers to the ensemble-λ.

Oldofredi and López (2020) do not argue that a ψ-epistemic wave-function is troublesome because of the PBR-theorem, that is, there are physical reasons to construe the wave-function differently; rather, they mention historical reasons for their take on the wave-function and the corresponding ontic state. They argue that Ballentine thought the wave-function in the statistical interpretation to provide a complete description of physical systems. Since any ψ-complete wave-function is necessarily ψ-ontic (see Lemma 6 in Harrigan and Spekkens, 2010, p. 133), the wave-function in the statistical interpretation is ψ-ontic.

Their historic argument is quite confusing. First, they present three sources in which Einstein explicitly argues that the wave-function has to be *incomplete* (to not violate locality).[14] Then they say that "Hence, it is fair to establish a strong theoretical continuity between Ballentine's presentation of the ensemble view and Einstein's interpretation of quantum mechanics." (p. 1330) That is, Ballentine agrees with Einstein on what the statistical interpretation tells us about quantum systems (see also Ballentine, 1972).[15] In particular, Ballentine is supposed to agree with Einstein that this theory is incomplete. As I discussed above, Ballentine sometimes appears to be indecisive as to whether his interpretation is complete or incomplete, but he certainly considers that his theory can be completed by another quantum theory. Oldofredi and López seem to suggest that Ballentine breaks with Einstein and advocates an interpretation that is complete necessitating a different ontic state λ that only refers to entire ensemble not to individual systems.

In my reading of Ballentine, I rather think that he considers his theory "complete for all practical purposes" to make successful predictions and to apply quantum mechanics without dealing with a mysterious metaphysics à la Copenhagen, but to get a truly complete theory that tells us what quantum systems really are, Ballentine seems to embrace such a completion of the statistical interpretation.

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[14] They are (i) Einstein's remarks at the 1927 Solvay conference (transcript in Bacciagaluppi and Valentini, 2009, pp. 440–442), (ii) Einstein's 1936 essay *Physics and Reality* (Einstein, 1936), and (iii) Einstein's reply to critics in his intellectual autobiography (Schilpp, 1949/1970).

[15] Fine (1993) also discusses how Einstein interpreted quantum mechanics.

Hence, it is Oldofredi and López's proposal of an ensemble-λ, which renders the statistical interpretation a complete theory, and therefore it is them who break with both Einstein's and Ballentine's view of how the statistical interpretation refers to the physical world.[16] I see the following problems with their suggestion. First, as I said, Einstein and Ballentine explicitly mention that their interpretation is incomplete. For all practical purposes, one can use the statistical interpretation to have a sufficiently clear picture of what quantum physics tells us about the world, but for a complete picture one would need to supplement this interpretation. Second, the statistical behavior of an ensemble is generated by its individual constituents. A natural question is how these constituents do that. How is the ensemble-λ related to the individual λs? Third, and connected to the previous point, even if the wave-function provides the complete description of an ensemble, it is unclear how the wave-function could not be ψ-statistical. Such a statistical interpretation of the wave-function would not provide the means to exclude that some particles can be swapped between two ensembles. One may respond that it is impossible to talk about the individual ontic states since all there is in this statistical interpretation is the ensemble-λ. This response is problematic, because the individual systems do exist, and it is unclear why we are not supposed to completely describe them or why the properties of the individual systems do not contribute to the complete description of the ensemble.

4.3 ψ-ontic and incomplete!

Because of the just mentioned conceptual problems with an ψ-ontic and complete wave-function for the statistical interpretation and the challenges of the PBR-theorem faced by a ψ-epistemic and incomplete wave-function, I conclude that a ψ-ontic and incomplete wave-function is the best option.

One may argue that a ψ-ontic and incomplete wave-function would restrict the relation between the ontic state λ and ψ more strictly than the formulation of the statistical interpretation justifies, because although the wave-function in the statistical interpretation only describes ensembles, we would need to believe in (a not yet specified) completion that would describe *individual systems*. This description of individual systems would make it impossible that an electron can be swapped from an x-up beam into a z-up beam. Even if the statistical interpretation does not seem to a priori prohibit such a swap of particles, this constraint follows from the wave-function being ψ-ontic.

[16] I also think that Oldofredi and López cite the wrong reason for why Harrigan and Spekkens consider the statistical interpretation to be ψ-epistemic:

> How do Harrigan and Spekkens reinforce their conclusion that the ensemble view is a ψ-epistemic model? They answer it by saying that the notion of 'ensemble" in Einstein's jargon is nothing but a way to talk about probabilities reflecting an observer's knowledge[.] (Oldofredi and López, 2020, p. 1328)

Harrigan and Spekkens do not make the mistake of identifying ψ-epistemic wave-functions with epistemic interpretations of the wave-function. Instead, they conclude that the wave-function in the statistical interpretation to be ψ-epistemic on two grounds. First, it is incomplete (according to Einstein and Ballentine). Second, Einstein gives an argument of associating two different wave-function to the same quantum system in his 1935 correspondence to Schrödinger.

That is, a particle is in a certain quantum state ψ not because it happens to be prepared to be part of an ensemble, but because properties of its ontic state λ make it to be in the quantum state ψ.

Although it is correct that the statistical interpretations allows for different ways the wave-function could be, I presented several arguments that uncover the problems of a ψ-epistemic or a complete wave-function. These arguments are similar in kind to arguments that have been made about the contextuality (Kochen and Specker, 1967) and non-locality of quantum mechanics (Bell, 1964/2004). The Kochen-Specker theorem concludes that a ψ-incomplete wave-function can only be supplemented by contextual variables (that is, variables that lead to different empirical results depending on how they are measured); Bell's theorem says that as long as a certain kind of statistical independence is fulfilled the theory cannot explain certain correlations by a local mechanism. So even if a certain interpretation of quantum mechanics is incomplete, one can discover certain hidden structure of the theory or certain features of a possible completion. The same is the case for my defense of a ψ-ontic and ψ-incomplete wave-function for the statistical interpretation.

This reading of the statistical interpretation would indeed break with Einstein's view of assigning several wave-functions to the same system but seems to back up my understanding of Ballentine of using the statistical interpretation for all practical purposes, while being open to a possible completion. Such an interpretation would also make the statistical interpretation fit into the ontological models framework without a revisionary interpretation of the ontic state λ.

5 Probabilities: Actual, Hypothetical, or Typical?

I now want to turn to a problem that Oldofredi and López (2020) correctly pointed out in the statistical interpretation: how should we properly interpret probabilities? Are they objective features of the ensemble, or do they reflect an agent's incomplete knowledge? I will conclude that the definition of probabilities in the statistical interpretation, although defined in terms of hypothetical ensembles, are in fact different from standard definitions in hypothetical frequentism. This definition turns out to be incomplete, and I offer a new definition that fixes this shortcoming.

5.1 Actual Frequentism

I agree with Oldofredi and López (2020) that the probabilities in the statistical interpretation are defined in a non-epistemic way, namely, as statistical properties of ensembles. Two main ideas have been discussed in the history of probability to make this more precise: actual frequentism and hypothetical frequentism. I do not think that it is correct that Harrigan and Spekkens identify actual frequentism as the underlying theory of probabilities in the statistical interpretation, as Oldofredi and López (2020, p. 1331) claim, "Indeed, the authors [namely, Harrigan and Spekkens] claim that the "ensemble talk" refers solely to the fact that probabilities should be interpreted as relative frequency in an actual, particular ensemble of systems." Harrigan and Spekkens are rather vague about the type of frequentism and only talk about "relative frequencies," whereas Oldofredi and López interpret "relative frequencies" to be actual frequencies.

In actual frequentism, probabilities are defined in the following way:

Actual Frequentism: The probability of an attribute A in a finite reference class B is the relative frequency of actual occurrences of A within B. (Hájek, 1996, p. 212)

Apart from the inherent problems of actual frequentism, as extensively discussed by Hájek (1996), actual frequentism clashes with the way probabilities are introduced in the statistical interpretation of quantum mechanics (and in quantum mechanics in general). Consider 100 electrons prepared in an x-up spin eigenstate, and we measure their spin in the z-direction. It is possible (and not too unrealistic) to find 60 of these electrons to hit the screen in the upper half, corresponding to a z-up state, and 40 in the lower half, corresponding to a z-down state. According to actual frequentism the probability of hitting the screen in the upper half (in the finite reference class of these very 100 electrons) would be 60% and in the lower half 40%. The Born rule, on the other hand, postulates a probability of 50% for hitting the screen in either half. The probabilities in actual frequentism would vary with the actual occurrences of events and would therefore be inconsistent with the Born rule.

5.2 Hypothetical Frequentism

Since actual frequentism does not work, one may define probabilities in the statistical interpretation of quantum mechanics as hypothetical frequencies, as Ballentine (1970, p. 361) does, "the relative frequency (or measure) of the various eigenvalues of the observable in the conceptual infinite ensemble of all possible outcomes in the identical experiments [...]". This is the way how hypothetical frequentism introduces probabilities (Hájek, 2009, p. 212):

Hypothetical Frequentism: The probability of an attribute A in a reference class B is p iff the limit of the relative frequency of occurrences of A within B would be p if B were infinite.

Hypothetical frequentism was an empiricist theory of probability that aimed to amend the problem of actual frequentism. The most famous developers of hypothetical frequentism were von Mises (1928/1957) and Reichenbach (1949/1971). They wanted to derive stable probabilities in terms of long-term frequencies, which are derived from the observable frequencies. Von Mises postulated two principles on the actual long-term frequencies (Rowbottom, 2015, p. 100–1):

1. *Law of Stability*: The relative frequencies of attributes in collectives[17] become increasingly stable as observations increase.
2. *Law of Randomness*: Collectives involve random sequences, in the sense that they contain no predictable patterns of attributes.

In von Mises theory, these laws are ad hoc, and only justified by yielding the right stable long-term frequencies. Von Mises argued that these principles are justified by induction.

[17]Collectives in von Mises' terminology are finite ensembles that fulfill certain requirements, like the law of stability and the law of randomness, to yield well-defined limiting frequencies.

Reichenbach (1949/1971) developed a similar theory to von Mises. He called his inductive law from actual to hypothetical frequencies the *Rule of Induction by Enumeration*. Starting with an infinite sequence of events A, he was interested in the relative frequency that some feature B occurs in this sequence. Let's say we have observed a sufficiently long finite sequence of events of length n. The frequency of feature B among the first n members of A is, say, $F^n(A, B) = {}^m\!/_n$. In order to infer the limiting frequency, the *Rule of Induction by Enumeration* needs to be applied: Given $F^n(A, B) = {}^m\!/_n$, to infer that $\lim_{n\to\infty} F^n(A, B) = {}^m\!/_n$ (Salmon, 1966, pp. 85–6). Reichenbach thought that it is possible to hit the "true probability" with a finite sequence. If that is the case, we can apply the Rule of Induction by Enumeration to get the same limiting frequency. This inference opens up an approximation procedure that we can derive just from the concept of the limit: for every ϵ there is an $N \in \mathbb{N}$ such that for all $n > N$, $F^n(A, B) \in [{}^m\!/_n - \epsilon, {}^m\!/_n + \epsilon]$. In words, for every small interval around the limit there is a sufficiently long sequence of events whose frequency will be in this interval. Or, we can approximate the limiting frequency to an arbitrary degree with a sufficiently long sequence of events.

Hypothetical frequentism in this form has many problems which makes it an untenable way of defining probabilities as argued by Hájek (2009) and La Caze (2016): the most notable are the reference class problem (see also Hájek, 2007), the problem of ascertainability (see also Salmon, 1966, section IV.2 & V.5), and the problem of the right ordering. Let me briefly introduce these three problems and explain how they make hypothetical frequentism unattractive:

1. *The reference class problem*: What is the correct infinite collection that gives rise to the right probability? More precisely, given a finite sequence of events $(x_1, \ldots, x_n)$, what is the appropriate infinite sequence $(y_1, y_2, \ldots)$ that we shall associate with $(x_1, \ldots, x_n)$ in order to assign the probability $p = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} y_i$ for some feature of $(x_1, \ldots, x_n)$?

2. *The problem of ascertainability*: How can we find out or ascertain in real life cases what the true probabilities are for a certain sequence of events? The problem is that the probabilities are defined in terms of unrealized hypothetical sequences, but we only observe finite sequences. Although Reichenbach defines the probabilities in a mathematically rigid way, it is unclear how to apply the limit to infer probabilities that we do not know yet.

3. *The problem of the right ordering*: Changing the order of the events in an infinite sequence can change the limit. One may be even able to change the ordering so that there is no limit at all. Both von Mises and Reichenbach were aware of this problem and acknowledged that their theory would not give a probability for these kinds of sequences.

The statistical interpretation introduces probabilities in the following way.

> Of primary importance is the assertion that a quantum state (pure or otherwise) represents an *ensemble* of similarly prepared systems. For example, the system may be a single electron. Then the ensemble will be the conceptual (infinite) set of all single electrons which have been subjected to some state preparation technique (to be specified for each state), generally by interaction with a suitable apparatus. (Ballentine, 1970, p. 361)

> In general quantum theory will not predict the result of a measurement of some observable R. But the probability of each possible result r_n [...] may be verified by repeating the state preparation and the measurement many times, and then constructing the statistical distribution of the results. As pointed out by Popper [...], one should distinguish between the *probability*, which is the relative frequency (or measure) of the various eigenvalues of the observable in the conceptual infinite ensemble of all possible outcomes of identical experiments (the sample space), and the *statistical frequency* of results in an actual sequence of experiments. The probabilities are properties of the state preparation method and are logically independent of the subsequent measurement, although the statistical frequencies of a long sequence of similar measurements (each preceded by state preparation) may be expected to approximate the probability distribution. (Ballentine, 1970, p. 361)[18]

In hypothetical frequentism, we start with an actually observed finite sequence, and define probabilities as the limit of these finite sequences. This is different in the statistical interpretation of quantum mechanics: probabilities are baked in a physical theory and defined in terms of (the modulus square of) the wave-function assigned to infinite ensembles. As in hypothetical frequentism, sufficiently long sequences approximate these probabilities.

Because the hypothetical frequencies in the statistical interpretation are provided by a physical theory, these probabilities are better justified than in standard accounts of hypothetical frequentism. In particular, the problems of the right reference class, of ascertainability, and the right ordering seem to be more comprehensively (but, in my opinion, still not completely) answered by the statistical interpretation. The problem of the right ordering is solved by the predictions of the physical theory: quantum mechanics predicts that the limiting frequencies are $|\psi|^2$. If the theory is well confirmed (as quantum mechanics is) then we do not have rational grounds to believe that there may be an ordering of an ensemble prepared in the same state that has a different limit or no limit at all. The problem of ascertainability also seems to be solved by the physical theory. Quantum mechanics specifies the probabilities for systems prepared in any quantum state ψ, and physicists know how to prepare these systems in the lab.

The reference class problem is only partially solved by the statistical interpretation. It does better than hypothetical frequentism because it specifies the ensemble property of the reference class, which is the wave-function. The wave-function is more than a shorthand for statistical behavior; it represents physical properties, and ensembles can be prepared to have these properties. Nevertheless, one issue in the statistical interpretation related to the reference class problem is the following. The ensemble mostly consists of hypothetical systems, which need not be identical with any actual system. What determines the properties of these systems? What determines the outcomes of

[18] Probabilities should answer (at least) one of two questions: (i) "What is there?", (ii) "What will happen?". An answer to the first question would result in a probability distribution over variables that exist independently of measurements. An answer to the second question is about what we observe. (I thank Charles Sebens for making this crucial distinction.) Ballentine introduces probabilities in the statistical interpretation in the second way, as (pragmatic) distributions for measurement outcomes. It seems to be a more difficult task to construe probabilities in the statistical interpretation in the first way, because the theory is incomplete, and a naive completion would run into non-contextuality issues (see, for instance, Norsen, 2017, Ch. 3.4).

hypothetical measurements on these systems? Are they determined by properties of some actual system? Are the statistical features of the (mostly imaginary) ensemble stipulated by stipulating a ψ-function?[19] It is unclear how the statistical interpretation of quantum mechanics can answer these questions. The main reason is that there is no description of individual systems that would anchor the statistical regularities of the ensemble, since the statistical interpretation only describes an ensemble as a whole.

That would not be a problem if the statistical interpretation is a ψ-incomplete theory, because a potential completion would provide the right answers. But for Oldofredi and López (2020) the statistical interpretation is a ψ-complete theory, and from the definition of probability applied to their interpretation we can again deduce that the theory has to be incomplete.

5.3 Typicality Frequentism

I will now present a new type of frequentism that grounds the hypothetical frequencies in the statistical interpretation on the behavior of individual systems. This theory of frequentism is particularly suitable for a completion of the statistical interpretation by the de Broglie–Bohm theory.

Typicality frequentism builds on hypothetical frequentism by solving the problems that plagued its predecessor theory (see, for instance, Bricmont, 2001; Goldstein, 2012; Maudlin, 2020; Hubert, 2021a). Here, the hypothetical frequencies are defined to be long-term typical frequencies *that arise from the initial conditions of the individual systems* within an ensemble (Hubert, 2021a, p. 11):

> *Typicality Frequentism:* Some state of affairs has probability p, if, according to a fundamental physical theory, the physical process X_k yielding this state of affairs is in principle infinitely repeatable and the instances of X_k are uncorrelated such that the frequency typically (that is, in almost all universes) approaches a unique limit p, that is, for all $\epsilon > 0$
> $\lim\limits_{N\to\infty} \mu\left(\left|\frac{1}{N}\sum_{k=1}^{N} X_k(x) - p\right| < \epsilon\right) = 1.$

In contrast to hypothetical frequentism, probabilities in typicality frequentism are not inductively defined from direct observations of finite sequences of events; rather, they are defined by a physical theory that describes the processes of the individual members of the ensemble. These individual processes give rise to a statistical pattern that a physical theory is able to extrapolate to sequences of arbitrary length. They key ingredient is a typicality measure μ on configuration space for quantum physics (or on phase space for classical physics), that weighs the different initial conditions of subsystems leading to the experimental results X_k. Then most initial conditions (according to the measure μ) result in a sequence of events whose actual frequency $\frac{1}{N}\sum_{k=1}^{N} X_k(x)$ approximate the probability p to arbitrary degree.

From the definition of probabilities in typicality frequentism, their empirical significance falls out immediately:

[19] I thank Christopher Hitchcock for raising these issues.

Empirical significance of p: The relative frequencies of finite sequences obey the restrictions given by the law of large numbers; that is, the observed frequencies of sufficiently long finite series typically lie in an interval $p \pm \epsilon$, where ϵ is a positive number approximately equal to 0.

As typicality frequentism requires a physical theory that describes the physical processes of individual systems (and not of the ensemble as a whole), it can more completely address the reference class problem than the statistical interpretation of quantum mechanics (as we discussed in section 5.2).

So embedding the statistical interpretation within typicality frequentism would require a detailed description of the physical state of individual systems; this again points to the incompleteness of the statistical interpretation. A possible ψ-ontic completion would be the de Broglie–Bohm pilot-wave theory (de Broglie, 1928; Bohm, 1952; Dürr and Teufel, 2009). In this theory, every particle is point-like, and its motion is given by a deterministic law that determines the location and the velocity of each particle for any time t. With this kind of mechanism for quantum processes (Glennan, 2017; Hubert, 2021b), one can *derive* the quantum probabilities in the spirit of typicality frequentism as typical long-term frequencies (see Dürr, Goldstein, and Zanghì, 1992, Dürr and Teufel, 2009, Ch. 8 , and Oldofredi et al., 2016). The result, which is a theorem in the de Broglie–Bohm theory rather than an axiom, is the Quantum Equilibrium Hypothesis:

Quantum Equilibrium Hypothesis: For an ensemble of identical systems, each having the wave function ψ, the typical empirical distribution of the configurations of the particles is given approximately by $\rho = |\psi|^2$.

This theorem is the "empirical significance of p" condition applied to the de Broglie–Bohm theory, and it completes the ensemble probabilities of the statistical interpretation. If an ensemble of systems is prepared in the quantum state ψ, the actual distribution of position measurements ρ is typically (that is, for almost all initial conditions) close to $|\psi|^2$, the probabilities determined by the de Broglie–Bohm theory (and, of course, quantum mechanics).[20] This distribution is derived from the complete state $\lambda_k = (x_t^k, \psi(x_t^k))$, where x_t^k is the actual configuration of the particles that constitute the kth quantum system in the ensemble of identically prepared systems.

Unlike, hypothetical frequentism, exceptions to the expected statistical pattern are explained in typicality frequentism: these atypical situations are derived from special initial conditions of the individual systems that lead to actual frequencies that wildly diverge from the hypothetical frequencies.

If the statistical interpretation of quantum mechanics is regarded to be ψ-ontic and ψ-incomplete, the de Broglie–Bohm theory is a natural completion that describes the states and behavior of individual quantum systems. By doing so, it also gives a more comprehensive picture of how probabilities arise in the statistical interpretation.

[20]The Quantum Equilibrium Hypothesis is formulated in terms of the distribution of positions, but it is valid for arbitrary observables.

6 Conclusion

Is the statistical interpretation of quantum mechanics ψ-epistemic? It could be. The core tenets of the interpretation do not settle the question as to whether the wave function is ψ-epistemic or ψ-ontic. That is left open, to be decided by deeper physics. If the statistical interpretation allows for the swap of some particles prepared with different wave-functions, the wave-function would be ψ-epistemic, but then it would be ruled out by the PBR theorem, unless one is willing to violate preparation independence and support a quantum theory that invokes some form of superdeterminism. The wave-function would be ψ-onic, if the statistical interpretation is completed by a quantum theory that describes the state of individual systems.

The way the statistical interpretation is formulated by Ballentine would paradoxically make both the ψ-epistemic and the ψ-ontic path viable. The underlying issue is that this interpretation in its very definition is incomplete. One can also spot this incompleteness in the way the statistical interpretation introduces probabilities as hypothetical frequencies. A possible completion of these hypothetical frequencies would be typicality frequentism, which would, however, demand that the statistical interpretation to be completed by a quantum theory like the de Broglie–Bohm pilot-wave theory. This would be a natural completion of a statistical interpretation that is ψ-ontic and ψ-incomplete.

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