Does the Einstein Algebra Formalism Favor Relationalism? A New Structural Comparison

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Abstract

In light of Chen (2024)'s recent objection to Rosenstock *et al.* (2015), this paper reconsiders the question "does the Einstein algebra formalism favor relationalism?". Following the structural comparison approach adopted by Rosenstock *et al.*, I propose a new formal criterion to investigate the question in place of the categorical criterion of theoretical equivalence, inspired by John Earman's program of Leibniz algebras. Based on the new criterion, the paper shows that the Einstein algebra formalism does not favor relationalism. It re-affirms Rosenstock *et al.*'s conclusion with a new technical result that is not subject to Chen's objection.

Contents

1 Introduction

The controversy of substantivalism versus relationalism, two metaphysical theses on the ontology of space and time, has a long history in philosophy of physics[1]. One important difference of the two positions is that relationalism is more parsimonious than substantivalism: relationalism doesn't posit the existence of space and time independent of material bodies, while substantivalism does. Which metaphysical view is favored by our physical theory of space and time? The standard mathematical formalism for the theory of space and time, the *standard manifold formalism*, starts with the mathematical structure of a smooth manifold, and then introduces fields on the manifold. For this reason, the standard manifold formalism seems to favor substantivalism. In contrast, the *Einstein algebra formalism* for the theory of space and time, formulated by Robert Geroch, defines fields without any background manifold-like mathematical structure. Does the Einstein algebra formalism favor relationalism? To answer this question, it is important to determine whether the Einstein algebra formalism is more parsimonious than the standard manifold formalism in the way that relationalism is more parsimonious than substantivalism. This calls for a structural comparison between the standard manifold formalism and the Einstein algebra formalism.

Philosophers have been working on the relationship between the Einstein algebra formalism and relationalism for several decades. John Earman's classic works propose a program of Leibniz algebras — an attempt of formulating a relationalist's formalism for the theory of space and time based on Einstein algebras (Earman, 1977, 1979, 1986, 1989). In response to Earman, Robert Rynasiewicz (1992) and Rosenstock *et al.* (2015) show various senses in which Einstein algebras are equivalent to relativistic spacetimes. They both point out that the structure of a smooth manifold and the relevant algebraic structure of the collection of smooth scalar fields on a manifold turn out to be mutually constructible from each other. Rosenstock *et al.* identify the relevant algebraic structure to be the smooth algebra introduced by Jet Nestruev (2003) and illustrate the equivalence of smooth manifolds and smooth algebras by establishing a categorical duality. They then show another categorical duality that holds for relativistic spacetimes and their version of Einstein algebras defined based on smooth algebras. Rosenstock *et al.* thus give a negative answer to the structural comparison question: Einstein algebras are not more structurally parsimonious than relativistic spacetimes, because the two are categorically dual to each other. Therefore, they suggest that insofar as one wants to associate the two formalisms with the two metaphysical views on the nature of space and time, the Einstein algebra formalism is as "substantivalist" as — and, for the same reason, as "relationalist" as — the standard manifold formalism (p315).

[1]For an overview, see Pooley (2013).

Nevertheless, Lu Chen (2024) recently points out that there is a gap in Rosenstock *et al.*'s argument. Rosenstock *et al.* assume that an Einstein algebra is a smooth algebra with a 4-dimensional Lorentizian metric and prove the categorical duality between Einstein algebras defined in this way and relativistic spacetimes. However, the definition of a smooth algebra postulates more structures than what Geroch (1972)'s formulation of Einstein algebras does. Chen criticizes this assumption of Rosenstock *et al.* and points out that the metaphysical question concerning the Einstein algebras originally defined by Geroch is in fact not addressed by their paper. This takes us back to the starting point. Motivated by Chen's criticism on Rosenstock *et al.*'s argument, this paper reconsiders the structural comparison of the standard manifold formalism and the Einstein algebra formalism, proposes a new criterion for the structural comparison, and eventually reaffirms Rosenstock *et al.*'s conclusion that the Einstein algebra formalism is as "substantivalist" as the smooth manifold formalism with a new result.

Here is the detailed plan of the paper. *Section 2* provides the technical background. I will first introduce the standard manifold formalism and the Einstein algebra formalism, and then present Rosenstock *et al.* (2015)'s categorical duality theorem of smooth manifolds and smooth algebras. After that, I will present Chen's criticism. To address the criticism, I take a step back to revisit Earman's program of Leibniz algebras in *Section 3*. After recapitulate Earman's main ideas, I formalize the program of Leibniz algebras with the tools of category theory. Inspired by this formalization, I propose a new formal criterion which states that the appropriately defined representation functor from the category of smooth manifolds to the category of algebraic models in consideration has to fail to be full and faithful for the algebraic formalism to be more parsimonious than the standard manifold formalism in the relationalist's sense, or in short, for the algebraic formalism to be "structurally relationalist". I show that the Einstein algebra formalism is not "structurally relationalist" based on this criterion (*Theorem 2*) and also re-cast Rosenstock *et al.*'s duality theorem as a corollary of this paper's main result. In *Section 4*, I further defend the proposed criterion for an algebraic formalism to be "structurally relationalist" with the example of Sikorski algebras (to be defined in *Section 4*) and differential spaces. Finally, the paper concludes with two messages: first, the Einstein algebra formalism does not favor relationalism; second, the proposed criterion can be a helpful replacement of the categorical criterion of theoretical equivalence for the substantivalism and relationalism debate.

2 Two Formalisms and a Duality Theorem

2.1 Two Formalisms

In the standard presentation of general relativity, one starts with the notion of *smooth manifolds*[2]. An n-dimensional smooth manifold (M, C) consists of a set M and an *atlas* C of *n-charts* on M, which defines a topology that is Hausdorff and second-countable, and introduces a smoothness structure on M, which allows us to identify *smooth maps* from M to another smooth manifold. Two manifolds (M, C) and (M', C') are called *diffeomorphic* to each other if there is a bijective smooth map from M to M' whose inverse is also smooth — this map is called a *diffeomorphism.* A *tensor field* on M is an assignment of a tensor to each point of the manifold. A *relativistic spacetime* is defined to be a Lorentzian manifold, which is a four-dimensional connected manifold with a smooth metric (tensor) field of Lorentzian signature. Relevant physical fields, such as the electromagnetic field, can be defined as tensor fields on the background spacetime. This is the *standard manifold formalism* for the theory of space and time.

On the other hand, alternative algebraic formalisms strive to define fields by their algebraic structures only, without referring to an underlying manifold. The mostly discussed algebraic formalism by philosophers of physics is the *Einstein algebras*, formulated by Robert Geroch (1972). Geroch observes that, for any smooth manifold M, the collection of all smooth scalar fields on M forms a commutative ring with pointwise addition and multiplication. Denote this commutative ring by $C^\infty(M)$. The collection of all constant functions on M forms a subring of $C^\infty(M)$ that is isomorphic to $\mathbb{R}$. He then illustrates how the mathematical notions, including tensor fields, needed for general relativity can be introduced by just constructing relevant algebraic structures based on $C^\infty(M)$, instead of defining them on the points of the underlying smooth manifold. Geroch therefore defines an *Einstein algebra* to consist of a commutative ring $\mathscr{F}$ which has a subring $\mathscr{R}$ isomorphic to $\mathbb{R}$ and a metric defined algebraically on $\mathscr{F}$ (Geroch, 1972, p274)[3].

For simplicity, this paper does not deal with the metrics in either the standard manifold formalism or the Einstein algebra formalism. This simplification should not compro-

[2]This brief presentation follows Malament (2012)'s and Wald (1984)'s definition of smooth manifolds. Refer to these textbooks for the full definitions of the notions presented in this paragraph.

[3]According to Geroch, we can define a contravariant vector field on M as a derivation on $C^\infty(M)$. The collection $\mathscr{D}$ of all smooth contravariant vector fields forms a module over the commutative ring $C^\infty(M)$. The dual module $\mathscr{D}^*$ of $\mathscr{D}$ is the collection of all smooth covariant vector fields. A metric g is then defined to be an isomorphism from $\mathscr{D}$ to $\mathscr{D}^*$ that satisfies the symmetry condition: for any ξ and η in $\mathscr{D}$, $g(\xi, \eta) = g(\eta, \xi)$, where $g(\xi, \eta)$ is defined to be $g(\xi)(\eta)$. This is equivalent to the symmetry condition in the standard manifold formalism. The fact that a metric is defined to be an isomorphism guarantees that it is non-degenerate. This algebraic definition of a metric therefore corresponds to the definition of a metric in the standard manifold formalism.

mise the argument of the paper, as Chen (2024)'s criticism doesn't concern the metric structure. Call a commutative ring which has a subring isomorphic to $\mathbb{R}$ an *Einstein ring*. The following discussion will focus on smooth manifolds and Einstein rings exclusively, from which I will draw conclusions about the standard manifold formalism and the Einstein algebra formalism.

A remarkable difference between the Einstein algebra formalism and the standard manifold formalism is that the former does not start with positing any background manifold-like structure for fields to be defined on. Do the two formalisms imply different ontological views on the nature of space and time? Substantivalism states that space and time exist independent of matter within them, while relationalism denies it. The way that the standard manifold formalism is constructed favors substantivalism, for the spacetime manifold is posited independently of and prior to fields. On the other hand, the Einstein algebra formalism appears to favor relationalism. This inference by inspection, however, is not sufficient. Rosenstock *et al.* (2015) claim that the standard manifold formalism and the Einstein algebra formalism are shown to be theoretically equivalent. The theoretical equivalence shows that "on a natural standard of comparison, the two theories have precisely the same mathematical structure — and thus, we claim, the same capacities to represent physical situations" (Rosenstock *et al.*, 2015, p315). Hence they claim that Einstein algebra formalism does not favor relationalism. This structural comparison approach of Rosenstock *et al.* connects the philosophical question concerning the nature of space and time to the formal frameworks of determining theoretical equivalence in physics. In addition to observing the way in which Einstein algebras are defined, the structural comparison approach uses formal tools to investigate whether a formalism is more parsimonious than another in terms of their mathematical structures. The structural parsimony of a formalism, compared to the standard manifold formalism, should correspond to the ontological parsimony of relationalism, compared to substantivalism. Rosenstock *et al.* infers from their equivalence result, or in other words, the lack of structural parsimony of the Einstein algebra formalism compared to the standard manifold formalism, to the conclusion that the Einstein algebra formalism cannot favor the ontologically parsimonious relationalism. I present their equivalence result in the next subsection.

2.2 The Duality Theorem

As surveyed by Weatherall (2019), philosophers of physics have widely discussed formal criteria of identifying theoretical equivalence. One prevalent criterion in the recent literature is the *categorical criterion of equivalence*. According to the categorical criterion of equivalence, two theories are equivalent if the categories of models of the two theories are equivalent as categories. This equivalence of categories is made precise by well-behaved functors between two categories[4]. Let **C** and **D** be two categories, and let F be a contravariant functor from **C** to **D**. F is said to be *faithful* if and only if for any two objects A and B in **C**, the induced map $(f : A \to B) \mapsto (F(f) : F(B) \to F(A))$ taking arrows from A to B in **C** to arrows from $F(B)$ to $F(A)$ in **D** is injective. F is said to be *full* if and only if for any two objects A and B in **C**, the induced map $(f : A \to B) \mapsto (F(f) : F(B) \to F(A))$ taking arrows from A to B in **C** to arrows from $F(B)$ to $F(A)$ in **D** is surjective. F is said to be *essentially surjective* if and only if for any D in **D**, there is an object A in **C** such that $F(A)$ is isomorphic to D in **D** — that is to say, there are arrows $f : F(A) \to D$ and $f^{-1} : D \to F(A)$ in **D** such that $f^{-1} \circ f = 1_{F(A)}$ and $f \circ f^{-1} = 1_D$. If F fails to be faithful, we say that F *forgets stuff*. If F fails to be full, we say that F *forgets structure*. If F is faithful, full, and essentially surjective, then F is said to realize a *categorical duality* of **C** and **D**, and the categories **C** and **D** are said to be *dual* to each other. Dual categories are *equivalent* to each other, as they can be viewed as "mirrored copies" of each other in the sense that the direction of their arrows is systematically reversed.

The category of models for smooth manifolds in the standard manifold formalism is defined by Rosenstock *et al.* (2015) as **SmoothMan** with the following:

- objects: smooth manifolds $M, N, \cdots$, and
- arrows: smooth maps $\varphi : M \to N$ where M and N are smooth manifolds.

On the algebraic side, Rosenstock *et al.* work with the notion of *smooth algebras* introduced by Jet Nestruev (2003). To give the definition of smooth algebras, some preliminaries are needed. We start with $\mathbb{R}$-*algebras*:

Definition 1 ($\mathbb{R}$-algebras)**.** *An* $\mathbb{R}$*-algebra* $\mathcal{A}$ *is a vector space over* $\mathbb{R}$ *with an additional associative and commutative vector multiplication and a multiplicative identity.*

It is not hard to see that an Einstein ring is an $\mathbb{R}$-algebra. We call the collection of $\mathbb{R}$-algebra homomorphisms — which preserve the vector space operations, products, and the multiplicative identity — from an $\mathbb{R}$-algebra $\mathcal{A}$ to $\mathbb{R}$ (which is also an $\mathbb{R}$-algebra) the *dual algebra* of $\mathcal{A}$, denoted by $|\mathcal{A}|$. Elements in the dual algebra of $\mathcal{A}$ are called *points* of the algebra $\mathcal{A}$. If $\mathcal{A}$ has only the zero element in the intersection of kernels of all the points of $\mathcal{A}$, then elements in $\mathcal{A}$ can be canonically identified with functions taking points of $\mathcal{A}$ to $\mathbb{R}$ by a bijective map. Such an algebra $\mathcal{A}$ is said to be *geometric* (Nestruev, 2003, p23). Define the coarsest topology on $|A|$ that makes every element of $\mathcal{A}$, canonically identified in the way described just now continuous. An $\mathbb{R}$-algebra $\mathcal{A}$ is said to be *complete* if it

[4]The following presentation of the categorical criterion of equivalence follows Weatherall (2019) and Rosenstock *et al.* (2015). It assumes familiarity with definitions of categories and functors, which can be found in standard textbooks on category theory such as (Awodey, 2010).

contains all the maps on $|\mathcal{A}|$ that are *locally equivalent* to elements of $\mathcal{A}$, in the sense that if $f : |\mathcal{A}| \to \mathbb{R}$ agrees with some $g \in \mathcal{A}$ restricting to some neighborhood of p for any $p \in |\mathcal{A}|$, then $f \in \mathcal{A}$ (Nestruev, 2003, p30-31). Restricting to a subset $U \subset |\mathcal{A}|$, the algebra $\mathcal{A}|_U$ is defined to be the $\mathbb{R}$-algebra containing all the functions $f : U \to \mathbb{R}$ that are locally equivalent to some element of $\mathcal{A}$. Now we give the definition of smooth algebras:

Definition 2 (Smooth algebras)**.** *A complete, geometric algebra $\mathcal{A}$ is called a smooth algebra if there is an at most countable open covering $\{U_k\}_{k\in\mathbb{N}}$ of $|\mathcal{A}|$ such that all the algebras $\mathcal{A}|_{U_k}, k \in \mathbb{N}$, are isomorphic to $C^\infty(\mathbb{R}^n)$ for some fixed natural number n.*

Given a smooth manifold M, the collection $C^\infty(M)$ of all its smooth scalar fields is not just an $\mathbb{R}$-algebra, but a smooth algebra (Nestruev, 2003, 7.5 & 7.6). The category of models for smooth algebras, **SmoothAlg**, is defined to consist of:

- objects: smooth algebras $\mathcal{A}, \mathcal{B}, \cdots$, and
- arrows: $\mathbb{R}$-algebra homomorphisms[5] $f : \mathcal{A} \to \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are smooth algebras.

The two categories **SmoothMan** and **SmoothAlg** turn out to be equivalent, as shown by the following duality theorem:

Theorem 1. ***SmoothMan*** *is dual to* ***SmoothAlg***. *(Rosenstock* et al.*, 2015, Theorem 3.5)*

According to the categorical criterion of theoretical equivalence, smooth manifolds and smooth algebras are equivalent. That is to say, the standard manifold formalism posits the same mathematical structure as the smooth algebra formalism does (Rosenstock *et al.*, 2015, p315). The smooth algebra formalism hence does not favor relationalism. Rosenstock *et al.* further identify an Einstein algebra as a 4-dimensional smooth algebra with additional structures defined on it (Rosenstock *et al.*, 2015, Section 4). Based on this assumption, they show that the Einstein algebra formalism and the relativistic spacetime formalism are equivalent, by establishing another categorical duality (Rosenstock *et al.*, 2015, Theorem 4.5), and conclude that the two formalisms "encode precisely the same physical facts about the world, in somewhat different languages" (Rosenstock *et al.*, 2015, p316). Rosenstock *et al.* therefore establish the following *conventional wisdom*: the Einstein algebra formalism is as "substantivalist" as the standard manifold formalism (ibid.). In other words, the conventional wisdom states that the Einstein algebra formalism does not favor relationalism.

[5] An $\mathbb{R}$-algebra homomorphism is a map that preserves the vector space operations, the product, and the multiplicative identity; a bijective algebra homomorphism is an $\mathbb{R}$-algebra isomorphism.

2.3 Chen's Criticism

In a recent paper, Chen advocates for the thesis of *algebraic relationalism*, which calls for "an algebraic implementation of relationalism", i.e. an algebraic formalism for the theory of space and time that is "not equivalent to substantivalism" (Chen, 2024, p2, p22). To defend the potential of this thesis, Chen challenges the conventional wisdom. She questions Rosenstock *et al.*'s inference from the categorical dualities they show to the conventional wisdom[6]. Specifically, Chen is skeptical of Rosenstock *et al.* use of smooth algebras as the basis of their discussion. Recall that in Definition 2, a smooth algebra is an $\mathbb{R}$-algebra that is complete, geometric, and smooth. Chen raises concerns for each of the three requirements. For the geometricity, she claims that this requirement rules out nilpotent algebras for no good reason other than that Rosenstock *et al.* want to establish a categorical duality:

> Why should we rule these out for algebraicism? No rationale is given by the authors other than the apparent reason that without this condition, we wouldn't be able to recover standard manifolds through categorical duality. (Chen, 2024, p10)

For the completeness, Chen argues that, to make sense of the idea of local equivalence of maps, we have to treat a neighborhood of $|\mathcal{A}|$ as a set of points on $|\mathcal{A}|$ so that restricting an $\mathbb{R}$-algebra homomorphism to a neighborhood makes sense. Hence she claims that "to require algebras to be complete, we make reference to geometric objects, which suggests this requirement as a disguised geometric discourse" (Chen, 2024, p10). Finally, for smoothness, Chen states that geometric concepts are directly invoked as the smoothness requirement is stated with topological vocabularies like open coverings. To sum up, Chen believes that the smooth algebra formalism is not properly algebraic, for the reasons that the only motivation to work with smooth algebras seems to be recovering the standard manifold formalism and that parts of its definition invoke geometric concepts.

Chen is correct to point out that a crucial motivation behind the smooth algebra formalism is to algebraically recover the standard manifold formalism, which is stated by its creator as the algebraic approach to define smooth manifolds:

> The two definitions of smooth manifold (in which the algebraic approach and the coordinate approach result) are of course equivalent. ... Essentially, this book is a detailed exposition of these two approaches to the notion of smooth manifold and their equivalence (Nestruev, 2003, p11)

[6] Chen also points to some philosophical shortcomings of the Einstein algebra formalism (Chen, 2024, p11-13) and to the advantages of other potential algebraic formalisms for the theory of space and time, in order to advocate for algebraic relationalism. These arguments are not a concern for the present paper, as our focus is the Einstein algebra formalism itself.

and recognized by Rosenstock *et al.* (2015). I also agree with Chen that smooth algebras have more stringent requirements than Einstein rings do — this applies to not only the geometricity but also the completeness and the smoothness. As stated before, it is not hard to see that an Einstein ring is an $\mathbb{R}$-algebra, but there is no indication in Geroch's original paper that Einstein rings must be geometric, complete, and smooth as in Definition 2. Therefore, it is unclear whether the categorical duality between smooth algebras and smooth manifolds shown in Theorem 1 is able to lead to any philosophical conclusion about the Einstein algebra formalism. If Rosenstock *et al.* were to establish a conventional wisdom concerning the smooth algebra formalism and relationalism, then they would have succeeded with their duality theorem. But in that case, as Chen points out, it would be redundant to do so, as smooth algebras are defined to be equivalent to smooth manifolds in the first place. This casts a doubt on the philosophical significance of Rosenstock *et al.*'s result, as it is unclear if Theorem 1 and its consequence imply anything about the Einstein algebra formalism at all. Therefore, we cannot trust the conventional wisdom anymore. The question "does the Einstein algebra formalism favor relationalism?" requires a new structural comparison between Einstein rings and smooth algebras in place of the categorical duality in Theorem 1.

As explained in the previous subsection, Rosenstock *et al.* (2015) apply the categorical criterion of equivalence directly to investigate whether or not the Einstein algebra formalism favors relationalism. While this approach is effective when a categorical equivalence between two formalisms can be shown, it falls short otherwise. If two formalisms are shown to posit the same mathematical structure, then one of them cannot favor an ontology of space and time per its mathematical structure while the other does not. Nevertheless, it is unclear whether theoretical equivalence is necessary for two formalisms to favor the same ontology in the first place. Recall that relationalism is a more parsimonious ontology than substantivalism. For an algebraic formalism to favor relationalism, it therefore should be more mathematically parsimonious than the smooth manifold formalism in the way that corresponds to the way that relationalism is a more parsimonious ontology than substantivalism. Call such an algebraic formalism "structurally relationalist". Being "structurally relationalist" is a necessary condition for an algebraic formalism to favor relationalism, as it formally tracks the parsimony aspect of relationalism. Instead of using the categorical criterion of equivalence, this paper will propose a formal criterion for the notion of "structural relationalist" and thus re-investigate whether the Einstein algebra formalism favors relationalism. This will be the task of the next section.

Before moving on to reconsider the relationship between the Einstein algebra formalism and relationalism, I shall briefly address the rest of Chen's criticism of smooth algebras, i.e. her concerns about the presence of so-called "geometric discourse" and geometric concepts in the definition of smooth algebras. Chen believes that they make smooth algebras not algebraic. It is not clear from her paper what definition of "algebraic-ness" she might have in mind. However, regardless of what it may be, it is not compatible with mathematical practice to regard any mathematical notion that involves concepts and terminologies like "neighborhood of points" or "open coverings" as not algebraic. To give a trivial example, we can always define a trivial topology on a given algebraic structure, which would allow us to meaningfully talk about neighborhood of points and open coverings. Numerous existing mathematical notions also deny the possibility of any clean division of "algebraic" versus "not algebraic" that Chen might have in mind. A C^* algebras is a Banach space; Boolean algebras admit stone space representations; Lie algebras are closely related to Lie groups and hence smooth manifolds. Therefore, even though I agree with the first half of Chen's criticism of smooth algebras being Rosenstock *et al.* (2015)'s technical basis of the conventional wisdom, I do not share her opinion that smooth algebras compromises on algebraic-ness.

3 Re-cast the Categorical Duality

In this section, I will reconsider the question "does the Einstein algebra formalism favor relationalism?" by proposing a new formal criterion for investigating this question in place of the categorical criterion of equivalence. I will start with illustrating and formalizing the ideas behind Earman's classic works of his program of Leibniz algebras, for it is the first introduction of Einstein algebras to the substantivalism and relationalism debate. The formalization results in the proposed formal criterion for an algebraic formalism to be "structurally relationalist". I will show that that, based on the new criterion, the conventional wisdom that the Einstein algebra formalism does not favor relationalism still holds, and the idea behind Rosenstock *et al.*'s categorical duality result can be preserved.

3.1 Einstein Algebras and the Program of Leibniz Algebras

In a series of publications, Earman (1977, 1979, 1986, 1989) introduces Einstein algebras as he explicates Leibniz' relational view on the spatio-temporal structure, which is extrapolated from Leibniz' writings on the nature of space and motion. Earman calls any model of the spatio-temporal structure of the form $\langle M, O_1, O_2, \cdots \rangle$ a *substantivalist world model*, where M is a smooth manifold and $O_i, i \in \mathbb{N}$ are geometric object fields on M.[7]

[7]The notation of a substantivalist world model varies in different pieces of Earman's writings. Here we follow (Earman, 1989, p171). He also calls certain substantivalist world models by certain names. For example,Earman (1977) defines a *Leibnizian pre-model*, which consists of a so-called *intermediate Leibnizian space-time* and a momentum field on the intermediate Leibnizian space-time (p100). The precise definition of the intermediate Leibnizian space-time does not concern us here. What's important

Given two substantivalist world models $\langle M, O_1, O_2, \cdots \rangle$ and $\langle M', O'_1, O'_2, \cdots \rangle$, if there is a diffeomorphism $\varphi : M \to M'$ such that $O'_i = \varphi^*(O_i), i \in \mathbb{N}$ — in Earman's words, φ^* denotes "dragging along" by the mapping φ (Earman, 1979, p268) — a relationalist like Leibniz must take the two substantivalist models as giving different descriptions of the same physical reality, instead of different physical realities. After all, space-time points are "descriptive fluff" (Earman, 1989, p170) for relationalists. In other words, one can say that these substantivalist world models are *Leibnizian equivalent*, which consequently gives rise to equivalence classes of substantivalist world models. According to Earman, an equivalence class of substantivalist world models corresponds to a single physical reality in the relationalist's sense[8]. However, it is not sufficient for relationalists to stop there, according to Earman. They need to complete an additional program, later called by Rynasiewicz (1992) "the program of Leibniz algebras", which consists of the following steps: (1) give a direct characterization of the relationalist's physical reality, i.e. the equivalence classes of substantivalist world models, without referencing to smooth manifolds. Earman (1977, 1979) calls such a direct characterization a *Leibniz world model*; (2) show that the laws of physics can be expressed directly in terms of Leibniz world models; (3) explain how Leibniz-equivalent substantivalist world models arise as different but equivalent representations of the same Leibniz world model; (4) show that the Leibniz world models are not subject to the hole argument.[9]

The motivation for the first two steps can be attributed to the substantivalist interpretation that Earman attaches to the standard manifold formalism and the Quine-Putnam style indispensability argument for substantivalism. One notable Quine-Putnam style indispensability argument is given by Hartry Field, who states that, since space-time points seems indispensable to positing physical fields, relationalists have to come up a different way of describing fields to avoid a realist's commitment to space-time points. We see from the following quote that Earman shares Field's worry:

> But drawing circles around groups of space-time models and labeling them equivalence classes does not show that there is a viable alternative to substantivalism. To show that one would have to show how to do all the physics we did before without treating the fields O_j as residing in M; in effect, one would have to show how to do differential geometry without the differential manifold. (Earman, 1986, p237)

for the purpose of this paper is that it is formulated in the standard manifold formalism, hence a substantivalist world model.

[8]See (Earman, 1977, p101), (Earman, 1979, p268), (Earman, 1986, p236-237), and (Earman, 1989, p171).

[9]My presentations of the program of Leibniz algebras is a combination of what Earman writes in (Earman, 1979) and (Earman, 1989). Step (4) is only explicit in (Earman, 1986, 1989), though a concern about indeterminism is visible in (Earman, 1977). See (Weatherall, 2020) for details.

But more importantly, it is also evident that Earman expects more from relationalists than merely coming up with an alternative formalism for the theory of space and time that is manifold-free. Immediately after the previous quote, he states the following:

> one would need to show how the old space-time models can be regarded as representations of the new models and prove that under the representation relation a single new model corresponds precisely to an equivalence class of old models. (Earman, 1986, p237)

That is to say, Leibnizian world models should correspond to Leibnizian-equivalent classes of substantivalist world models in the sense that a *representation* relation[10] between substantivalist world models and Leibniz world models should be defined, and that Leibniz-equivalent substantivalist world models represent one and the same Leibniz world models. This requires the step (3) of the program. In other words, Leibniz world models are expected to get rid of the "descriptive fluff" in the substantivalist world models, via the "many-to-one" representation relation from substantivalist world models to Leibniz world models.

Finally, Earman claims that "the desire for the possibility of determinism . . . provides an independent motivation for a program like the above" (Earman, 1989, p172)[11]. The worry that the standard manifold formalism and the substantivalist interpretation of it imply the impossibility of determinism is most notably spelled out in (Earman & Norton, 1987), in the form of the hole argument. To briefly summarize, the hole argument is based on the *Hole Corollary* that's proven in the same paper, which states that given a substantivalist world model $\langle M, O_1, O_2, \cdots \rangle$ and a neighborhood $U \subseteq M$, there exist arbitrarily many $\langle M, O'_1, O'_2, \cdots \rangle$'s which differ from $\langle M, O_1, O_2, \cdots \rangle$ only within U and is identical to $\langle M, O_1, O_2, \cdots \rangle$ on the boundary and outside of U. The neighborhood U is called a hole. If we place a hole U in the future of a time slice, then, for substantivalists, all the history up to that time slice is unable to determine the future, as all substantivalist world models which only differ within U match the history up to the given time slice. As Earman & Norton assume that distinct substantivalist world models represent distinct substantivalist physical realities, they argue that substantivalists have to deny any possibility of determinism for the theory of space and time[12]. The hole argument leads Earman to believe that the standard manifold formalism for the theory of space and time has excess structures and to suggest the Einstein algebra formalism as a suitable

[10]Earman also calls a *realization* relation in several other places.

[11]Similar remarks can be found in (Earman, 1977) and (Earman, 1986).

[12]The hole argument receives great attention in the subsequent philosophical literature, including objections by Weatherall (2018) and Halvorson & Manchak (2024). This paper does not discuss the validity of the hole argument. The purpose of presenting the hole argument here is to explain one motivation Earman's program of Leibniz algebras, for which it plays an important role (Weatherall, 2020, p81).

modification of it, as (Weatherall, 2020, p86) points out. That is to say, Earman expects a reletionalist's formalism to get rid of the excess structures in substantivalist world models, which he believes are shown to exist by the hole argument. This is what step (4) is for.

To sum up, the program of Leibniz algebras aims to ward off the indispensability argument and, more importantly, to get rid of excess structures in the standard manifold formalism. More specifically, the reasons why Earman believes that there are excess structures in the standard manifold formalism are that, first, relationalists should regard Leibniz-equivalent substantivalist world models as mere different descriptions of the same physical reality, and second, he believes that the standard manifold formalism is subject to the hole argument. As a result, Leibniz algebras are expected to get rid of excess structures by accomplishing step (3) and (4).

3.2 Formalizing the Program of Leibniz Algebras

In this subsection, I formalize the program of Leibniz algebras with category theory and propose a formal criterion for determining "structurally relationalist" algebraic formalism based on the formalization. I should make one caveat before proceeding. Earman does not consider Leibniz/Einstein algebras to be fully relationalist. He argues that Leibniz algebras do not escape the objection based on principle of sufficient reason (PSR), as there is no reason why God would choose to actualize one Leibniz algebra instead of another isomorphic but distinct Leibniz algebra (Earman, 1986, p239). For this reason, he calls Leibniz algebras to be only "first-degree non-substantivalist" but substantivalist at a deeper level. Can we still use the program of Leibniz algebras as a guide to formally characterizing the notion of "structurally relationalist"?[13] I say yes, for the following two reasons. First, the reason why Earman claims that Leibniz algebras are substantivalist at a deeper level is that they cannot bypass the PSR objection. This worry does not concern the objective of this paper, which focuses on the ontological parsimony aspect of relationalism. It is also a controversial whether the multiplicity of Leibniz algebras should be a concern for the substantivalism versus relationalism controversy, as Weatherall (2018) and Bradley & Weatherall (2022) argue that it is an unavoidable consequence of constructing mathematical objects in set theory. Second, Earman also gives some positive remarks indicating that the mathematical structure of Leibniz algebras satisfies the following two necessary conditions for a fully relationalist formalism:

> Leibniz algebras provide a solution to the problem of characterizing the structure common to a Leibniz-equivalence class of substantival models and the

[13] I am grateful to an anonymous referee for raising this point.

solutions eschews substantivalism in the form of space-time points (Earman, 1989, p192-193)

Therefore, even though Leibniz algebras are not considered to be a thoroughgoing relationalist formalism, it is legitimate to take Earman's ideas underlying the program of Leibniz algebras for the purpose of characterizing "structurally relationalist" formalisms.

The program of Leibniz algebras concerns two kinds of world models: substantivalist world models and Leibniz world models. Substantivalist world models are connected by smooth maps that preserve geometric fields. Similarly, Leibniz world models are connected by algebraic homomorphisms that preserve fields defined algebraically. Suppose that we can describe the two kinds of world models by two categories — call them the *substantivalist category* and the *Leibniz category* respectively. Since substantivalist world models are expected to represent Leibniz world models, in the language of category theory, we can think of the representation relation as a functor to be defined from the substantivalist category to the Leibniz category. Call this the *representation functor*. If the representation functor, appropriately defined, can show the ways in which Earman expects Leibniz world models to get rid of the excess structures in substantivalist world models, then we can say that the Leibniz world models, i.e. the Leibniz algebras, are "structurally relationalist".

Returning to smooth manifolds and Einstein rings, the category **SmoothMan** takes the place of the substantivalist category here. For the algebraic side, we define the category **EinRings** as follows:

- objects: Einstein rings $\mathscr{E}, \mathscr{F}, \cdots$, and
- arrows: Einstein ring homomorphisms $h : \mathscr{E} \to \mathscr{F}$ where $\mathscr{E}$ and $\mathscr{F}$ are Einstein ringsn, and let $\mathscr{R}_{\mathscr{E}}$ be the subring of $\mathscr{E}$ that is isomorphic to $\mathbb{R}$ and $\mathscr{R}_{\mathscr{F}}$ be the subring of $\mathscr{F}$ that is isomorphic to $\mathbb{R}$, then $h|_{\mathscr{E}}$ is a ring isomorphism from $\mathscr{R}_{\mathscr{E}}$ to $\mathscr{R}_{\mathscr{F}}$.

We denote a representation functor from **SmoothMan** to **EinRings** by R. Step (3) and (4) of the program of Leibniz algebras can be interpreted as the following two expectations of the behavior of the representation functor R.

Therefore, in terms of **SmoothMan** and **EinRings**, diffeomorphic smooth manifolds should be mapped by the representation functor R to one and the same Einstein ring. Moreover, as a Leibniz algebra is expected by Earman to directly characterize what a Leibniz-equivalent class of substantivalist world models represent, it should characterize the mathematical structure that is shared by substantivalist world models in one Leibniz-equivalence class. Hence a structure-preserving map defined between two Leibniz algebras should be expected to preserve "less structure" than a structure-preserving map of substantivalist world models does. As a result, for a morphism from one Leibniz algebra to another Leibniz algebra, there might not be a corresponding morphism from the the substantivalist world model that represents the first Leibniz algebra to the substantivalist world model that represents the second. Therefore, in terms of **SmoothMan** and **EinRing**, we can interpret that the program of Leibniz algebras expects the representation functor R to be *not full*. Figure 1 illustrates this interpretation. Let diffeomorphic

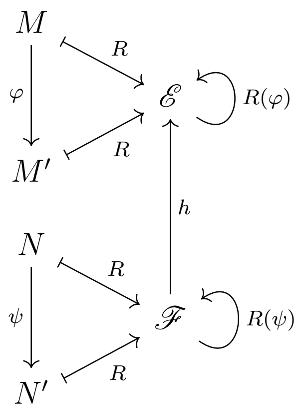

Figure 1:

smooth manifolds M and M' be mapped by the representation functor R to Einstein ring $\mathscr{E}$, the diffeomoprhism $\varphi : M \to M'$ be mapped to $R(\varphi) : \mathscr{E} \to \mathscr{E}$. Another pair of diffeomorphic smooth manifolds N and N' are mapped by the representation functor R to Einstein ring $\mathscr{F}$, the diffeomorphism $\psi : N \to N'$ is mapped to $R(\psi) : \mathscr{F} \to \mathscr{F}$. Let $h : \mathscr{F} \to \mathscr{E}$ be an arrow in **EinRings**. Based on the interpretation, the program of Leibniz algebras expects that there is not always a morphism from M to N (or, similarly, a morphism from M' to N') that is mapped to h by R. That is to say, the fullness of the functor is not expected to hold.

As Earman believes that the hole argument is an important indicator that the substantivalist world models have excess structures, the Einstein algebra formalism is expected to get rid of the excess structures that lead to the hole argument. Step (4) of the program of Leibniz algebras is stipulated for this purpose. Recall that, according to the hole argument, the impossibility of determinism is a result of the existence of substantivalist world models that are identical except within a neighborhood of the spacetime manifold (the hole). These substantivalist world models have the same smooth manifold, and they can be derived from one another with a diffeomorphism of the smooth manifold to itself which leaves all the fields in the substantivalist world model unchanged except within the hole. Call these diffeomorphisms the *hole diffeomorphism*[14]. As a relationalist formal-

[14]Recently, Halvorson & Manchak (2024) argue that no hole isomorphism exists if it is required to be a isometry of relativistic spacetimes. Since this paper focuses on only the smooth manifold structure, we

ism is expected by Earman to be not subject to the hole argument, we can understand that as expecting no counterpart of hole diffeomorphisms for Einstein algebras to exist. In terms of **SmoothMan** and **EinRings**, we can interpret this expectation to be that the representation functor R would fail to be faithful. To illustrate, suppose that, as in

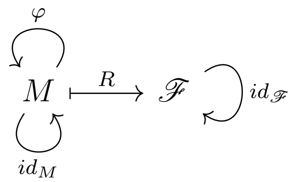

Figure 2:

Figure 2, a smooth manifold M is mapped to an Einstein ring $\mathscr{F}$ by the representation functor R. Let $\varphi : M \to M$ be a hole diffeomorphism and id_M be the identity arrow of M. Since the program expects Einstein algebras to be free from the hole argument, the hole diffeomorphism φ would not be mapped by R to an isomorphism of $\mathscr{F}$ that has the potential to give rise to a hole argument on Einstein algebras. Instead, we expect φ to be mapped to the identity arrow $id_{\mathscr{F}}$ of $\mathscr{F}$. In that case, $R(id_M) = id_{\mathscr{F}} = R(\varphi)$. Therefore, the representation functor R would be not faithful.

The formalization in this subsection shows that, according to the program of Leibniz algebras, the representation functor R from **SmoothMan** to **EinRings** has to fail to be full and fail to be faithful for the Einstein algebra formalism to be "structurally relationalist". This criterion can be applied to any other algebraic formalism by substituting **EinRings** with the category of models of the algebraic formalism in consideration. This criterion is in a sense stricter than simply requiring the two categories to be not dual to each other, as it requires the duality to be spoiled specifically by the representation functor being not full and not faithful. The requirement of essential surjectivity in the categorical criterion of theoretical equivalence is therefore not relevant, according to the program of Leibniz algebras. In the next section, I defend this claim independently of the program of Leibniz algebras. For the rest of this section, we apply the new criterion to **SmoothMan** and **EinRings**.

3.3 Re-cast the Duality

Now we are ready to answer the question this paper started with: "foes the Einstein algebra formalism favor relationalism?"? We adopt the formal criterion that the representation functor R from **SmoothMan** to **EinRings** has to fail to be full and faithful

do not comment on the implications of Halvorson & Manchak (2024)'s result here.

for a positive answer. We define the representation functor R from **SmoothMan** to **EinRings** as follows:

- Given a smooth manifold M, $R(M) = C^\infty(M)$.
- Given a smooth map $\varphi : M \to N$ from manifold M to manifold N, $R(\varphi) = \hat{\varphi} : C^\infty(N) \to C^\infty(M)$ where $\hat{\varphi}(f) = f \circ \varphi$ for all $f \in C^\infty(N)$ is an Einstein ring homomorphism[15].

The justification of this definition is straightforward. A smooth manifold M must be able to represent the Einstein ring $C^\infty(M)$. The mapping of arrows also makes intuitive sense. Then the following theorem is shown to be true:

Theorem 2. *The representation functor R : **SmoothMan** $\to$ **EinRings** is full and faithful.*

Proof. To see that the representation functor R is faithful, let M and N be two smooth manifolds and $\varphi : M \to N$ and $\psi : M \to N$ be smooth maps such that $\varphi \neq \psi$. Then there must be some $p \in M$ such that $\varphi(p) \neq \psi(p)$. Since N is a Hausdorff manifold, there is a smooth map $f \in C^\infty(N)$ such that $f(\varphi(p)) \neq f(\psi(p))$. Hence $R(\varphi)(f) = \hat{\varphi}(f) \neq \hat{\psi}(f) = R(\psi)(f)$, which implies that $R(\varphi) \neq R(\psi)$.

To see that the representation functor F is full, let M and N be two smooth manifolds and $\theta : C^\infty(N) \to C^\infty(M)$ be an Einstein ring homomorphism. By (Nestruev, 2003, Theorem7.7), there is an atlas that can be defined on $|C^\infty(M)|$ such that $|C^\infty(M)|$ is a smooth manifold and $C^\infty(|C^\infty(M)|)$ is isomorphic to $C^\infty(M)$. Similarly, $|C^\infty(N)|$ can be endowed with a smoothness structure such that $C^\infty(|C^\infty(N)|)$ is isomorphic to $C^\infty(N)$. Furthermore, there is a bijective homeomorphism between $\theta_M : M \to |C^\infty(M)|$ defined by

$$\theta_M(p)(f) = f(p)$$

for all $p \in M$ and $f \in C^\infty(M)$, given by (Nestruev, 2003, Theorem 7.2). By (Rosenstock *et al.*, 2015, Theorem 3.5). θ_M is a diffeomorphism. Similarly, there is a diffeomorphism $\theta_N : N \to |C^\infty(N)|$. Define $|\theta| : |C^\infty(M)| \to |C^\infty(N)|$ as follows:

$$|\theta|(\gamma)(\theta_N(k)) = \theta(k)(\theta_M^{-1}(\gamma))$$

for all $\gamma \in |C^\infty(M)|$ and $k \in C^\infty(N)$. $|\theta|$ is a smooth map by the proof of (Rosenstock *et al.*, 2015, Lemma 3.4). Therefore, $\theta_N^{-1} \circ |\theta| \circ \theta_M : M \to N$ is a smooth map, i.e. an arrow in **SmoothMan**. Finally, $R(\theta_N^{-1} \circ |\theta| \circ \theta_M) = \theta$ because

$$R(\theta_N^{-1} \circ |\theta| \circ \theta_M)(g)(p) = g(\theta_N^{-1}(|\theta|(\theta_M(p)))) = \theta(g)(p)$$

[15](Rosenstock *et al.*, 2015, Lemma 3.3) shows that, given a smooth map $\varphi : M \to N$, $\hat{\varphi} : C^\infty(N) \to C^\infty(M)$ defined as $\hat{\varphi}(f) = f \circ \varphi$ for all $f \in C^\infty(N)$ is an $\mathbb{R}$-algebra homomorphism. It is not hard to see that $\hat{\varphi}$ is also an Einstein ring homomorphism, as every Einstein ring is an $\mathbb{R}$-algebra.

for any $g \in C^\infty(N), p \in M$. □

Therefore, we conclude that the Einstein algebra formalism is not "structurally relationalist" and therefore does not favor relationalism. Theorem 2 does not involve any additional algebraic structure than the Einstein ring structure assumed in Geroch's original formulation of Einstein algebras. Hence it resolves Chen's objection that the conventional wisdom is supported by Rosenstock *et al.* (2015)'s duality theorem that involves smooth algebras instead of the less stringent Einstein rings. We therefore re-affirm the metaphysical status of the Einstein algebra formalism that Rosenstock *et al.* establish, which is that it is as "relationalist", and equivalently as "substantivalist", as the standard manifold formalism. The conventional wisdom established by Rosenstock *et al.* (2015) is still true, though the technical result that supports it takes a different form.

Moreover, I note that the representation functor R is in fact almost the same as one contravariant functor that Rosenstock *et al.* use to prove the duality of **SmoothMan** and **SmoothAlg** (p312). Their duality theorem can therefore be re-stated as a corollary of Theorem 2:

Corollary. *The image of the representation functor* R : ***SmoothMan*** $\rightarrow$ ***EinRings*** *is equivalent to* ***SmoothAlg****.*

Proof. Since smooth algebras are Einstein rings and $\mathbb{R}$-algebra homomorphisms are Einstein ring homomorphisms, Theorem 2 shows that the image of the representation functor R is a full and faithful sub-category of **SmoothAlg**, in the sense that the inclusion functor from R[**SmoothMan**] to **SmoothAlg** which maps all objects and arrows to themselves is full and faithful. To see the equivalence, we need to show that the inclusion functor from R[**SmoothMan**] to **SmoothAlg** is essentially surjective. As pointed out by (Nestruev, 2003, Theorem 7.7), for any smooth algebra $\mathcal{A}$, there exists a smooth atlas on its dual space $|\mathcal{A}| =: M$ such that $\mathcal{A}$ is isomorphic to $C^\infty(M)$ as smooth algebras. That is to say, for any object $\mathcal{A}$ of **SmoothAlg**, there must be some object $R(M)$ of R[**SmoothMan**] where M is an object of **SmoothMan**, such that $\mathcal{A}$ is isomorphic to $R(M)$ in **SmoothAlg**. Therefore, the inclusion functor from R[**SmoothMan**] to **SmoothAlg** is essentially surjective. □

The corollary shows that the fundamental insight of Rosenstock *et al.* (2015)'s duality theorem is not as irrelevant to the Einstein algebra formalism as Chen's objection might have indicated. There are significant overlaps between the categorical criterion of equivalence, which Rosenstock *et al.* work with, and the stricter criterion this paper adopts to investigate the metaphysical stance of the Einstein algebra formalism. For one contravariant functor they use to establish categorical duality between **SmoothMan** and **SmoothAlg** is almost the same as the representation functor R, what Rosenstock *et al.* show is effectively a part of the picture that this paper presents.

4 The Irrelevance of Essential Surjectivity

We have given an answer to the question this paper started with. A crucial component of it is the new formal criterion for an algebraic formalism to be "structurally relationalist". So far the only justification I have given for this formal criterion is the formalization of Earman's program of Leibniz algebras. In this section, I give another justification with the example of Sikorski algebras and differential spaces. The example shows that an algebraic formalism that spoils only the essential surjectivity of the representation functor from **SmoothMan** to a category of its models does not move us closer to relationalism. This is because a failure of essential surjectivity has nothing to do with a failure of geometric reconstruction. The irrelevance of essential surjectivity supports the new formal criterion which requires only faithfulness and fullness of the representation functor.

To motivate the example, recall the categorical duality of **SmoothMan** and **SmoothAlg**. We are interested in the following question: if an algebraic formalism breaks the duality by only spoiling the essential surjectivity of the contravariant functor from **SmoothMan** to **SoomthAlg**, will that make the algebraic formalism favor relationalism? We therefore relax the smoothness condition of smooth algebras and consider a less stringently defined algebraic structure, the Sikorski algebra, defined as follows:

Definition 3 (C^∞-closure)**.** *A geometric $\mathbb{R}$-algebra $\mathcal{A}$ is said to be C^∞-closed if for any finite collection of its elements $f_1, \cdots, f_k \in \mathcal{A}$ and any $g \in C^\infty(\mathbb{R}^n)$ for some n, there exists an element $f \in \mathcal{A}$ such that*

$$f(a) = g \circ (f_1(a), \cdots, f_k(a)), \text{ for all } a \in |\mathcal{A}|.$$

Note that the function $f \in \mathcal{A}$ here is uniquely determined, since $\mathcal{A}$ is geometric. (Nestruev, 2003, p33)

Definition 4 (Sikorski algebras)**.** *We call an $\mathbb{R}$-algebra $\mathcal{A}$ a Sikorski algebra if $\mathcal{A}$ is geometric, complete, and C^∞-closed.*

We note that the Sikorski algebras bear great similarities to C^∞-rings in the contemporary mathematics literature (for example, see (Joyce, 2012)). The rationale behind naming this algebraic structure "Sikorski algebras" is that, as Gruszczak *et al.* (1988) and Heller (1991) point out, Sikorski (1971) was the first who discussed this kind of algebraic structure. To see that Sikorski algebras have weaker requirements than smooth algebras do, we note the following facts:

Lemma 1. *Every smooth algebra is a Sikorski algebra. (Nestruev, 2003, Proposition 4.4)*

Lemma 2. *Not every Sikorski algebra is a smooth algebra.*

Proof. The collection of all real-valued continuous functions on $\mathbb{R}$, $C^0(\mathbb{R})$, is a Sikorski algebra but not a smooth algebra. $\square$

We define the category **SikorskiAlg** as consisting of the following:

- objects: Sikorski algebras $\mathcal{A}, \mathcal{B}, \cdots$, and
- arrows: $\mathbb{R}$-algebra homomorphisms $i : \mathcal{A} \to \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are Sikorski algebras.

Similarly, we define a representation functor R' from **SmoothMan** to **SikorskiAlg**, based on how Rosenstock *et al.* (2015) define the contravariant functor from **SmoothMan** to **SmoothAlg** (p312), as follows:

- Given a smooth manifold M, $R'(M) = C^\infty(M)$.
- Given a smooth map $\varphi : M \to N$ from manifold M to manifold N, $R(\varphi) = \hat{\varphi} : C^\infty(N) \to C^\infty(M)$ where $\hat{\varphi}(f) = f \circ \varphi$ for all $f \in C^\infty(N)$ is an $\mathbb{R}$-algebra homomorphism.

The representation functor R' also bears a great similarity to the representation functor R defined in the previous section. Moving from **SmoothAlg** to **SikorskiAlg** only spoils the essential surjectivity, as shown by the following theorem:

Theorem 3. *The representation functor R' from **SmoothMan** to **SikorskiAlg** is faithful and full, but not essentially surjective.*

Proof. By Lemma 1, a similar reasoning to the proof of Theorem 2 shows that the representation functor R' is faithful and full. Lemma 2 implies that the functor R' is not essentially surjective, for the reasons that $\mathbb{R}$-algebra isomorphisms, i.e. isomorphism arrows in **SikorskiAlg**, preserve smoothness of $\mathbb{R}$-algebras and that all objects in the range of R' are smooth algebras. $\square$

Theorem 3 shows that Sikorski algebras satisfy the antecedent of the question we are investigating. In the rest of this section, I show that the Sikorski algebra formalism does not favor relationalism, independent of the argument based on the program of Leibniz algebras in section 3. To establish this conclusion, we first note that the categorical duality of **SmoothMan** and **SmoothAlg** shows that the standard manifold formalism and the smooth algebra formalism are structurally equivalent. Neither of them favors relationalism more than the other formalism does, for the reason that:

> Both encode precisely the same physical facts about the world, in somewhat different languages. (Rosenstock *et al.*, 2015, p315-316)

We follow this line of thought and show that Sikorski algebras encode precisely the same physical facts about the world as a generalized geometric structure from smooth manifolds — *differential spaces* (see Sikorski (1971), Heller (1992, 1991), and Gruszczak *et al.* (1988)) — by establish a categorical duality. We introduce the following definitions associated with differential spaces:

Definition 5 (Differential spaces)**.** *Let M be a set and D be a collection of $\mathbb{R}$-valued maps on M. Let τ_D be the coarsest topology on M such that all functions in D are continuous and that the following two hold:*

1. *if $f : M \to \mathbb{R}$ is a map such that, for every $p \in M$, there is a neighborhood U of p in the topological space (M, τ_D) and a map $g \in D$ such that $f|_U = g|_U$, then $f \in D$.*
2. *for any $n \in \mathbb{N}$ and any function $\omega \in C^\infty(\mathbb{R}^n)$, $f_1, \cdots, f_n \in D$ implies $\omega \circ (f_1, \cdots, f_n) \in D$.*

Then we say that D is a differential structure on M and that (M, D) is a differential space. (Gruszczak et al.*, 1988, Definition 3.1-3.4)*

Definition 6 (D-maps between differential spaces)**.** *Let (M, D_M) and (N, D_N) be differential spaces. A map $\Phi : M \to N$ is called a d-map from (M, D_M) to (N, D_N) if $h \circ \Phi \in D_M$, for every $h \in D_N$.*

Definition 7 (D-diffeomorphisms between differential spaces)**.** *Let (M, D_M) and (N, D_N) be differential spaces. A bijective map $\Phi : M \to N$ is called a d-diffeomorphism of (M, D_M) and (N, D_N) if $h \circ \Phi \in D_M$ and $g \circ \Phi^{-1} \in D_N$, for every $h \in D_N$ and $g \in D_M$. (Gruszczak* et al.*, 1988, Definition 4.1)*

According to Gruszczak *et al.* (1988) and Heller (1991), differential spaces are generalizations of smooth manifolds. An additional condition can be imposed on differential spaces to turn them into *differential manifolds*, which are equivalent to smooth manifolds defined in the standard way[16]. Generalizing from **SmoothMan**, we define a new category **D-Spaces** as consisting of the following:

- objects: differential spaces $(M, D_M), (N, D_N), \cdots$, and
- arrows: d-maps $\Phi : M \to N$ where (M, D_M) and (N, D_N) are differential spaces.

Finally, we show that Sikorski algebras and differential spaces have the same mathematical structure by establishing the following categorical duality:

Theorem 4. ***SikorskiAlg*** *is dual to* ***D-Spaces****.*

Proof. Define a contravariant functor G : **SikorskiAlg** $\to$ **D-Spaces** as follows:

- For each object $\mathcal{A}$, $G(\mathcal{A}) = (|\mathcal{A}|, \mathcal{A})$.
- For each arrow $i : \mathcal{A} \to \mathcal{B}$ between Sikorski algebras $\mathcal{A}$ and $\mathcal{B}$, $G(i) : |\mathcal{B}| \to |\mathcal{A}|$ is defined as $G(i)(p)(x) := i(x)(p)$, for all $p \in |\mathcal{B}|, x \in \mathcal{A}$.

We show that the functor G is well-defined. First, it is not hard to see that $G(\mathcal{A})$ is a differential space, since $\mathcal{A}$ is a complete and C^∞-closed $\mathbb{R}$-algebra. Secondly, we show that, for each $\mathbb{R}$-algebra homomorphism $i : \mathcal{A} \to \mathcal{B}$ between Sikorski algebras $\mathcal{A}$ and $\mathcal{B}$, $G(i) : |\mathcal{B}| \to |\mathcal{A}|$ defined as above is a d-map. For any element x of $\mathcal{A}$, we need to show

[16] For a technical explanation of this equivalence, see (Gruszczak *et al.*, 1988, section 4).

that $x \circ G(i) : |\mathcal{B}| \to \mathbb{R}$ is a continuous map. Since a composition of continuous maps is continuous, it is sufficient to show that $G(i)$ is a continuous map from $|\mathcal{B}|$ to $|\mathcal{A}|$. To see that, note that for every $U \subseteq G(i)[|\mathcal{B}|]$ such that $U = x^{-1}(U_{\mathbb{R}})$ for some $x \in \mathcal{A}$, $U_{\mathbb{R}}$ an open set in $\mathbb{R}$, i.e. U is open given the subspace topology of τ_A on $G(i)[|\mathcal{B}|]$, we have

$$[G(i)]^{-1}[U] = [i(x)]^{-1}[U_{\mathbb{R}}],$$

which is an open set in $|\mathcal{B}|$ given τ_B. This is because that $[G(i)]^{-1}[U] = \{p \in |\mathcal{B}| : G(i)(p)(x) \in U\} = \{p \in |\mathcal{B}| : i(x)(p) \in U_{\mathbb{R}}\}$ and $i(x) \in \mathcal{B}$ by definition. Therefore $G(i)$ is continuous. Therefore, $G(i)$ defined as above is a d-map between $|\mathcal{B}|$ and $|\mathcal{A}|$. The contravariant functor G : **SikorskiAlg** $\to$ **D-Spaces** is well-defined.

Define a contravariant functor H : **D-Spaces** $\to$ **SikorskiAlg** as follows:

- For each object (M, D_M) in **D-Spaces**, $H((M, D_M)) = D_M$.
- For each arrow $\Phi : M \to N$, i.e. a d-map between (M, D_M) and (N, D_N), $H(\Phi) : D_N \to D_M$ is defined by $H(\Phi)(k) = k \circ \Phi$ for any $k \in D_N$.

To see that H is a well-defined contravariant functor, first note that for a differential space (M, D_M), there is a one-to-one correspondence between M and $|D_M|$, characterized by the map $\eta_M : M \to |D_M|$ defined as follows:

$$\eta_M(p)(f) = f(p)$$

for all $p \in M$, $f \in D_M$. Therefore, if (M, D_M) is a differential space, then D_M is a Sikorski algebra, by definition. Next, it is not hard to see that, for each arrow $\Phi : M \to N$, $H(\Phi) : D_N \to D_M$ is defined by $H(\Phi)(k) = k \circ \Phi$ for any $k \in D_N$ is an $\mathbb{R}$-algebra homomorphism, for it preserves the vector space operations, the product, and the multiplicative identity. Therefore, H is a well-defined contravariant functor.

Now we show that GH : **D-Spaces** $\to$ **D-Spaces** is naturally isomorphic to $1_{\textbf{D-Spaces}}$. We define a family of maps associated with GH, between objects (M, D_M) and $GH((M, D_M)) = (|D_M|, D_M)$ in **D-Spaces**, as follows:

$$\eta_M : M \to |D_M| \quad \text{s.t.} \;\; \eta_M(p)(f) = f(p)$$

for all $p \in M$, $f \in D_M$. That η_M is a d-diffeomorphism follows from the fact that the two differential spaces have the same differential structure and that it is surjective by definition. Therefore, GH is naturally isomorphic to $1_{\textbf{D-Spaces}}$. One the other hand, we can see that HG : **SikorskiAlg** $\to$ **SikorskiAlg** is the same as $1_{\textbf{SikorskiAlg}}$ by definition of G and H. Therefore, **D-Spaces** is dual to **SikorskiAlg**. $\square$

As Sikorski algebras and differential spaces have the same mathematical structure, to say that the mathematical structure of Sikorski algebras favors relationalism is equivalent to saying that the mathematical structure of differential spaces favors relationalism. Nevertheless, it doesn't make sense to say that the differential space formalism makes a weaker ontological commitment regarding the independent existence of space and time than the standard manifold formalism. If the standard manifold formalism is considered to represent space and time with its mathematical structure, then the differential space formalism merely disagrees with it about what mathematical structure represents space and time. As differential spaces are generalizations of smooth manifolds, from a substantivalist's perspective, the differential space formalism simply stipulates a different fundamental spatio-temporal structure than the fundamental spatio-temporal structure that the standard manifold formalism stipulates. The structural equivalence of Sikorski algebras and differential spaces therefore leads to the conclusion that the Sikorski algebra formalism does not favor relationalism. If anything, the Sikorski algebra formalism, and equivalently the differential space formalism, just provides a different substantivalist picture of space and time from the one that the standard manifold formalism provides.

To sum up, the example of Sikorski algebras shows that the essential surjectivity of the representation functor is not relevant to the question of whether an algebraic formalism is "structurally relationalist". It is because that, despite that the representation functor R' from **SmoothMan** to **SikorskiAlg** is not essentially surjective, Sikorski algebras cannot be reasonably viewed to favor relationalism due to the theoretical equivalence between Sikorski algebras and differential spaces. This example justifies the proposed formal criterion which states that an algebraic formalism for the theory of space and time is "structurally relationalist" if the appropriately defined representation functor from **SmoothMan** to the category of algebraic models fails to be faithful and full, regardless of whether it is essentially surjective or not.

5 Conclusion

To conclude, this paper begins with presenting Rosenstock *et al.* (2015)'s duality theorem and the conventional wisdom stating that the Einstein algebra formalism does not favor relationalism. Then I presented Chen (2024)'s recent objection to the conventional wisdom. Motivated by Chen's objection, I re-visited Earman's classic works on the program of Leibniz algebras and proposed a formal criterion for determining whether an algebraic formalism is "structurally relationalist", which is a necessary condition for it to favor relationalism. Based on this criterion, I showed that the conventional wisdom is still true, though supported by a different technical result. Finally, I provided another justification of the proposed criterion, which provided additional support to re-affirming the conventional wisdom.

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